Multiparameter quantum groups at roots of unity

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Goal: Study multiparametric deformations of universal enveloping algebras of simple Lie algebras at roots of unity.

- Relation with classical geometric objects.
- Frobenius maps.
- Small quantum groups = Frobenius-Lusztig kernels.
- Relation with multiparametric deformations of function algebras of connected, simply connected and simple Lie groups.
- New examples of Hopf algebras as quantum subgroups.

We fix

- $\ell \in \mathbb{N}_+$, $I := \{1, \dots, \ell\}$.
- $A := \big(a_{ij}\big)_{i,j \in I}$ symmetrizable Cartan matrix of finite type.
  $D := \big(d_i\, \delta_{ij}\big)_{i,j \in I}$ diagonal matrix such that DA is symmetric
- $\mathfrak{g}$ complex simple Lie algebra associated with A.
- Φ finite root system of $\mathfrak{g}$, with basis $\Pi = \big\{ \alpha_i \,\big|\, i \in I \big\}$.
  $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ root lattice, $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ weight lattice.
- $\mathbb{Z}$–bilineal form $(\ ,\ ) : P \times Q \longrightarrow \mathbb{Z}$, given by
  $(\omega_i, \alpha_j) := d_i \delta_{ij}$ for all $i, j \in I$.
- $\mathbb{k}$ a field of characteristic zero.
- $q \in \mathbb{k}$, with $q \neq 0$, $q_i := q^{d_i} \neq \pm 1$, for all $i \in I$.

Definition (Drinfeld-Jimbo)

The *quantized universal enveloping algebra* of $\mathfrak{g}$ is the algebra $U_q(\mathfrak{g})$ generated by $K_i^{\pm 1}$, E_i, F_i with $1 \leq i \leq \ell$ satisfying

$$K_iK_j = K_jK_i, \qquad K_i^{\pm 1}K_i^{\mp 1} = 1,$$
$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j,$$
$$K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k}E_jE_i^k = 0 \text{ for all } i \neq j,$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k}F_jF_i^k = 0 \text{ for all } i \neq j,$$

Moreover, $U_q(\mathfrak{g}_A)$ is a (usual) Hopf algebra with:

$$
\begin{aligned}
\Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, & \epsilon(K_i^{\pm 1}) &= 1, & \mathcal{S}(K_i^{\pm 1}) &= K_i^{\mp 1}, \\
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \epsilon(E_i) &= 0, & \mathcal{S}(E_i) &= -K_i^{-1} E_i, \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, & \epsilon(F_i) &= 0, & \mathcal{S}(F_i) &= -F_i K_i.
\end{aligned}
$$

Fix $\mathbf{q} := \left(q_{ij}\right)_{i,j \in I}$, with coefficients in $\Bbbk^\times$

We say that $\mathbf{q} := \left(q_{ij}\right)_{i,j \in I}$ is of *Cartan type* if there exists a (indecomposable) generalized Cartan matrix $A = \left(a_{ij}\right)_{i,j \in I}$ such that

$$q_{ij}\, q_{ji} \;=\; q_{ii}^{a_{ij}} \qquad\qquad \forall \;\; i, j \in I$$

If A is indecomposable, there is $j_0 \in I$ such that $q_{ii} = q_{j_0 j_0}^{e_i}$ for some $e_i \in \mathbb{N}$, for all $i \in I$.

We assume $\Bbbk$ has a square root of $q_{j_0 j_0}$, and write $q_{j_0} := \sqrt{q_{j_0 j_0}}$. We also fix $q_i := q_{j_0}^{e_i}$ for all $i \in I$.

Definition (Hu, Pei, Rosso)

$U_{\mathbf{q}}(\mathfrak{g}_D)$ *is the* $\Bbbk$*-algebra generated by* E_i , F_i , $K_i^{\pm 1}$, $L_i^{\pm 1}$ *satisfying*

$$(a) \qquad K_i^{\pm 1} L_j^{\pm 1} = L_j^{\pm 1} K_i^{\pm 1} \;, \qquad K_i^{\pm 1} K_i^{\mp 1} = 1 = L_i^{\pm 1} L_i^{\mp 1}$$

$$(b) \qquad K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1} \;, \qquad L_i^{\pm 1} L_j^{\pm 1} = L_j^{\pm 1} L_i^{\pm 1}$$

$$(c) \qquad K_i\, E_j\, K_i^{-1} = q_{ij}\, E_j \;, \qquad L_i\, E_j\, L_i^{-1} = q_{ji}^{-1}\, E_j$$

$$(d) \qquad K_i\, F_j\, K_i^{-1} = q_{ij}^{-1}\, F_j \;, \qquad L_i\, F_j\, L_i^{-1} = q_{ji}\, F_j$$

$$(e) \qquad [E_i, F_j] = \delta_{i,j}\, q_{ii}\, \frac{K_i - L_i}{q_{ii} - 1}$$

$$(f) \qquad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\binom{k}{2}}\, q_{ij}^k\, E_i^{1-a_{ij}-k} E_j\, E_i^k = 0 \; (i \neq j)$$

$$(g) \qquad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\binom{k}{2}}\, q_{ij}^k\, F_i^{1-a_{ij}-k} F_j\, F_i^k = 0 \; (i \neq j)$$

$U_{\mathbf{q}}(\mathfrak{g}_D)$ is a Hopf algebra with

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i \ , & \epsilon(E_i) &= 0 \ , & \mathcal{S}(E_i) &= -K_i^{-1} E_i \\
\Delta(F_i) &= F_i \otimes L_i + 1 \otimes F_i \ , & \epsilon(F_i) &= 0 \ , & \mathcal{S}(F_i) &= -F_i L_i^{-1} \\
\Delta\big(K_i^{\pm 1}\big) &= K_i^{\pm 1} \otimes K_i^{\pm 1} \ , & \epsilon\big(K_i^{\pm 1}\big) &= 1 \ , & \mathcal{S}\big(K_i^{\pm 1}\big) &= K_i^{\mp 1} \\
\Delta\big(L_i^{\pm 1}\big) &= L_i^{\pm 1} \otimes L_i^{\pm 1} \ , & \epsilon\big(L_i^{\pm 1}\big) &= 1 \ , & \mathcal{S}\big(L_i^{\pm 1}\big) &= L_i^{\mp 1}
\end{aligned}$$

Integral type: $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ is of *integral type* if it is of Cartan type and there exist $b_{ij} \in \mathbb{Z}$ such that $q_{ij} = q^{\,b_{ij}}$ for all $i, j \in I$; where $b_{ii} = 2\,d_i$ y $b_{ij} + b_{ji} = 2\,d_i\,a_{ij}$ $(\,i, j \in I\,)$, with $q = q_{j_0}$.

Strong integral type: $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ is of integral type and $b_{ij} \in d_i\,\mathbb{Z} \bigcap d_j\,\mathbb{Z}$ for all $i, j \in I$. That is, there exists $t_{ij}^{+}, t_{ij}^{-} \in \mathbb{Z}$ such that $q_{ij} = q^{\,d_i t_{ij}^{+}} = q^{\,d_j t_{ij}^{-}}$ for all $i, j \in I$. We assume also $t_{ii}^{\pm} = 2 = a_{ii}$ and $t_{ij}^{+} + t_{ji}^{-} = 2\,a_{ij}$, for all $i, j \in I$.

Standard type: Given $q \in \Bbbk^{\times}$ one defines

$$\check{q}_{ij} \;:=\; q^{\,d_i a_{ij}} \qquad \forall \;\, i, j \in I$$

"standard" $\Longrightarrow$ *"strong integral"* $\Longrightarrow$ *"integral"* $\Longrightarrow$ *"Cartan"*

Theorem (Hu, Pei, Rosso)

Let $\mathbf{q} = (q_{ij})_{i,j \in I}$ *be a multiparameter of Cartan type. Then there exists a Hopf 2-cocycle on* $U_{\check{\mathbf{q}}}(\mathfrak{g}_D)$ *such that*

$$U_{\mathbf{q}}(\mathfrak{g}_D) \;\cong\; \big(\, U_{\check{\mathbf{q}}}(\mathfrak{g}_D)\big)_{\sigma}$$

Taking in $\mathfrak{b}_+$ and $\mathfrak{b}_-$ the generators $\mathrm{e}_i, \mathrm{h}_i^+$ ($i \in I$) and $\mathrm{f}_i, \mathrm{h}_i^-$ ($i \in I$), one may construct the *Manin double* $\mathfrak{g}_D = \mathfrak{b}_+ \oplus \mathfrak{b}_-$.

For $A^{-1} = \left(a'_{ij}\right)_{i,j \in I}$, we define $\mathrm{t}_i^{\pm} = \pm \sum_{k \in I} a'_{ik} \mathrm{h}_i^{\pm}$. In particular, $\left[\mathrm{t}_i^{\pm}, \mathrm{e}_j\right] = \pm \delta_{ij} \mathrm{e}_j$, $\left[\mathrm{t}_i^{\pm}, \mathrm{f}_j\right] = \mp \delta_{ij} \mathrm{f}_j$ for $i, j \in I$.

Let $\mathbf{q} := \left(q_{ij} = q^{b_{ij}} \right)_{i,j \in I}$ be of *integral type*. Define the $\mathbb{Z}$-Lie subalgebra $\dot{\mathfrak{g}}_{D,\mathbf{q}}$ of $\mathfrak{g}_D$ as the one generated by e_i , f_i , h_i , $\dot{\mathrm{k}}_i := \sum_{j \in I} b_{ij} \mathrm{t}_j^+$ y $\dot{\mathrm{l}}_i := \sum_{j \in I} b_{ji} \mathrm{t}_j^-$.

It is a Lie bialgebra with cobracket

$$\delta(\mathrm{e}_i) = \dot{\mathrm{k}}_i \otimes \mathrm{e}_i - \mathrm{e}_i \otimes \dot{\mathrm{k}}_i \quad , \qquad \delta(\mathrm{f}_i) = \mathrm{f}_i \otimes \dot{\mathrm{l}}_i - \dot{\mathrm{l}}_i \otimes \mathrm{f}_i$$

$$\delta(\dot{\mathrm{k}}_i) = 0 \quad , \qquad \delta(\dot{\mathrm{l}}_i) = 0 \quad , \qquad \delta(\mathrm{h}_i) = 0$$

Let $\mathbf{q} := \left(q_{ij} = q^{d_i t_{ij}^+} = q^{d_j t_{ij}^-} \right)_{i,j \in I}$ be of *strong integral type*.
Define the $\mathbb{Z}$-Lie subalgebra $\mathfrak{g}_{D,\mathbf{q}}$ of $\mathfrak{g}_D$ as the one generated by
$\mathrm{e}_i\,,\ \mathrm{f}_i\,,\ \mathrm{k}_i := \sum_{j \in I} t_{ij}^+\,\mathrm{t}_j^+\,,\ \mathrm{l}_i := \sum_{j \in I} t_{ji}^-\,\mathrm{t}_j^-\,,\ \mathrm{h}_i$

It is a Lie bialgebra with cobracket

$$\delta(\mathrm{e}_i) = d_i(\mathrm{k}_i \otimes \mathrm{e}_i - \mathrm{e}_i \otimes \mathrm{k}_i) \quad , \qquad \delta(\mathrm{f}_i) = d_i(\mathrm{f}_i \otimes \mathrm{l}_i - \mathrm{l}_i \otimes \mathrm{f}_i)$$
$$\delta(\mathrm{k}_i) = 0 \quad , \qquad \delta(\mathrm{l}_i) = 0 \quad , \qquad \delta(\mathrm{h}_i) = 0$$

Rmk: Note that $\dot{\mathfrak{g}}_{D,\mathbf{q}}$ y $\mathfrak{g}_{D,\mathbf{q}}$ are $\mathbb{Z}$–integer forms of $\mathfrak{g}_D$.

The Lie algebra structure does not depend on $\mathbf{q}$*, but the co-structure does!*

Let $\mathcal{F}_{\mathbf{q}}$ be the subfield of $\Bbbk$ generated by the $q_{ij}^{\pm 1}$'s together with $q^{\pm 1}$. As a base ring we fix the subring $\mathcal{R}_{\mathbf{q}}$ of $\Bbbk$ generated by the $q_{ij}^{\pm 1}$'s and $q^{\pm 1}$.

q–divided powers : Given $i \in I$ y $X_i \in \{E_i, F_i\}$, we define $X_i^{(n)} := X_i^n \Big/ (n)_{q_{ii}}!$ in $U_{\mathbf{q}}(\mathfrak{g}_D)$, where $(n)_{q_{ii}} := \frac{q_{ii}^n - 1}{q_{ii} - 1}$

For $p \in \Bbbk$ not a root of 1, $n \in \mathbb{N}$ and $H \in \{K_i, L_i \mid i \in I\}$ define

$$\binom{H}{n}_p := \prod_{s=1}^{n} \frac{p^{1-s} H - 1}{p^s - 1}$$

Let $\mathbf{q}$ be of integral type and consider $U_{\mathbf{q}}(\mathfrak{g}_D)$ the MpQG defined over $\mathcal{F}_{\mathbf{q}}$. We define the $\mathcal{R}_{\mathbf{q}}$-subalgebras of $U_{\mathbf{q}}(\mathfrak{g}_D)$:

$$\dot{\widehat{U}}_{\mathbf{q}}^{0} := \left\langle L_i^{\pm 1}, K_i^{\pm 1}, \binom{L_i}{n}_q, \binom{K_i}{n}_q \right\rangle_{i \in I, n \in \mathbb{N}},$$

$$\dot{\widehat{U}}_{\mathbf{q}} = \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}_D) := \left\langle \dot{\widehat{U}}_{\mathbf{q}}^{0} \bigcup \left\{ F_i^{(n)}, E_i^{(n)} \right\}_{i \in I, n \in \mathbb{N}} \right\rangle$$

Assume we defined in $U_{\mathbf{q}}(\mathfrak{g}_D)$ root vectors E_α, F_α for all $\alpha \in \Phi^+$. Consider the *renormalizations*

$$\overline{E}_\alpha := \left(q_{\alpha\,\alpha} - 1 \right) E_\alpha, \qquad \overline{F}_\alpha := \left(q_{\alpha\,\alpha} - 1 \right) F_\alpha \qquad \forall\ \alpha \in \Phi^+,$$

where $q_{\alpha\,\alpha} := \prod_{i,j \in I} q_{ij}^{a_i a_j}$ for $\alpha = \sum_{i \in I} a_i \, \alpha_i \in Q$.

Define in $U_{\mathbf{q}}(\mathfrak{g}_D)$ the $\mathcal{R}_{\mathbf{q}}$-subalgebra:

$$\widetilde{U}_{\mathbf{q}} := \left\langle \overline{F}_\alpha \,,\, L_i^{\pm 1},\, K_i^{\pm 1},\, \overline{E}_\alpha \right\rangle_{i \in I,\, \alpha \in \Phi^+}.$$

Define the especializations at 1:

$$\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g}_D) \;:=\; \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}_D) \Big/ (q-1)\, \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}_D),$$

$$\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}_D) \;:=\; \widetilde{U}_{\mathbf{q}}(\mathfrak{g}_D) \Big/ (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}_D).$$

Theorem

$\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g}_D)$ is a (cocommutative) co-Poisson Hopf algebra isomorphic to $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{D,\mathbf{q}}\big)$.

Theorem

$\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}_D)$ is isomorphic to the function algebra $\mathcal{O}\big(\widetilde{G}^*_{D,\mathbf{q}}\big)$ of a Poisson group scheme over $\mathbb{Z}$ with cotangent Lie bialgebra $\tilde{\mathfrak{g}}_{D,\mathbf{q}}$.

Let $p_\ell(x)$ be the ℓ–the cyclotomic polynomial in $\mathbb{Z}[x]$ and ε a primitive ℓ–th root of unity.
Define the especializations at ε by

$$\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \ := \ \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}_D) \Big/ p_\ell(q)\, \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}_D)$$

$$\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \ := \ \widetilde{U}_{\mathbf{q}}(\mathfrak{g}_D) \Big/ p_\ell(q)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}_D)$$

Write $\mathcal{R}_{\mathbf{q},\varepsilon} = \mathcal{R}_{\mathbf{q}}/(p_\ell(x))$, $\mathcal{R}_{\mathbf{q},1} := \mathcal{R}_{\mathbf{q}} \Big/ (q-1)\,\mathcal{R}_{\mathbf{q}}$.

Theorem

There exists a Hopf algebra epimorphism

$$\widehat{\dot{Fr}}_\ell \,:\, \widehat{\dot{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \twoheadrightarrow \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widehat{\dot{U}}_{\mathbf{q},1}(\mathfrak{g}_D) \cong \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{D,\mathbf{q}}\big)$$

Theorem

There exists a Hopf algebra monomorphism

$$\widetilde{Fr}_\ell \,:\, \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}_D) \longhookrightarrow \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$$

The small quantum group $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ is defined as the $\mathcal{R}_{\mathbf{q},\varepsilon}$–subalgebra of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$, given by

$$\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} = \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) := \left\langle F_i^{(n)}, L_i^{\pm 1}, \binom{L_i}{n}_{\varepsilon}, K_i^{\pm 1}, \binom{K_i}{n}_{\varepsilon}, E_i^{(n)} \right\rangle_{i \in I}^{0 \leq n \lneq \ell}$$

Theorem

Consider the scalar extension to $\mathbb{Q}_\varepsilon$ of the Frobenius map $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \xrightarrow{\dot{\widehat{Fr}}_\ell} U_{\mathbb{Q}_\varepsilon}\big(\dot{\mathfrak{g}}_{D,\mathbf{q}}\big)$. Then

$$1 \longrightarrow \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} \xrightarrow{\iota} \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \xrightarrow{\dot{\widehat{Fr}}_\ell} U_{\mathbb{Q}_\varepsilon}\big(\dot{\mathfrak{g}}_{D,\mathbf{q}}\big) \longrightarrow 1$$

is an exact sequence of $\mathbb{Q}_\varepsilon$–Hopf algebras which is cleft.

Let Z_0 be the $\mathcal{R}_{\mathbf{q},\varepsilon}$–subalgebra of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$ given by

$$Z_0 \;:=\; \Big\langle\, \overline{F}_\alpha^{\,\ell} \,,\; L_i^{\pm\ell} \,,\; K_i^{\pm\ell} \,,\; \overline{E}_\alpha^{\,\ell} \,\Big\rangle_{\alpha\in Q\,,\,i\in I}$$

Proposition (Angiono)

(a) *Z_0 is a central Hopf subalgebra of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$.*

(b) *$\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$ is a free Z_0–module of rank $\ell^{\,dim(\mathfrak{g}_D)}$.*

The unrestricted small quantum groups is defined as

$$\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon} \;:=\; \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \;:=\; \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \Big/ \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)\, {Z_0}^{+}$$

Theorem

Let $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}_D) \xhookrightarrow{\widetilde{Fr}_\ell} \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$ *be the scalar extension to* $\mathbb{Q}_\varepsilon$ *of the unrestricted Frobenius map; here* $\mathrm{Im}\,\widetilde{Fr}_\ell = Z_0$*. Then*

(*a*)

$$1 \longrightarrow \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}_D) \xrightarrow{\widetilde{Fr}_\ell} \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \xrightarrow{\pi} \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \to 1$$

is an exact sequence of $\mathbb{Q}_\varepsilon$*-Hopf algebras, which is cleft.*

(*b*) $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D) \cong \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g}_D)$ *as Hopf algebras over* $\mathbb{Q}_\varepsilon$ *.*