Minimal vertex separators and new characterizations for dually chordal graphs

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Quick definitions

A *uv*-separator of $G$ is a set $S \subseteq V(G)$ such that $G - S$ is disconnected, with $u$ and $v$ in two different connected components. It is **minimal** if no smaller subset has the same property. $S(G)$ will denote the family of minimal vertex separators of $G$.

A **complete** is a set of pairwise adjacent vertices. A **clique** is a maximal complete. $C(G)$ denotes the family of cliques of $G$.

A family of sets is **Helly** if the intersection of all the elements of any subfamily of pairwise intersecting sets is not empty.

If $C(G)$ is a Helly family, we say that $G$ is a **Helly** graph.
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A **chord** of a cycle is an edge joining nonconsecutive vertices of the cycle.

**Chordal** graphs are those without chordless cycles of length at least four.

The **intersection graph** of a family $F$ of sets, $L(F)$, has these sets as vertices and two of them are adjacent if they are not disjoint.

We refer to $L(C(G))$ as the **clique graph** of $G$ or just $K(G)$.
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Dually chordal graphs

Definitions

Given a graph $G$, $w$ is a maximum neighbor of $v$ if $N^2[v] \subseteq N[w]$.

$v_1v_2...v_n$ is a maximum neighborhood ordering of $G$ if $v_i$ has a maximum neighbor in $G[\{v_i, ..., v_n\}]$.

We call dually chordal to any graph with a maximum neighborhood ordering.
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Example

1723645 is a maximum neighborhood ordering.
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Other characterizations

- There exists a spanning tree $T$ such that every clique induces a subtree.
- There exists a spanning tree $T$ such that, $\forall v \in V(G)$, $N[v]$ induces a subtree.

Any tree $T$ with these characteristics is given the name of compatible tree.
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Separators and neighborhoods

**Theorem**
Let $u$ and $v$ be two nonadjacent vertices of a dually chordal graph $G$. Then there is a vertex $w$, $w \neq u$ and $w \neq v$, such that $N[w] - \{u, v\}$ is a $uv$-separator.

**Sketch of proof**
Take $T$ compatible with $G$ and let $w$ be an inner vertex of the path in $T$ from $u$ to $v$. 
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**Theorem**

Let $G$ be a dually chordal graph. Then every minimal vertex separator of $G$ is contained in the neighborhood of a vertex if and only if every chordless cycle of $G$ is in the neighborhood of some vertex.

What happened with the graph of the example?
Theorem

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New characterizations

It is possible to prove that...

- If a tree is compatible with $G$ then each minimal vertex separator induces a subtree.
- If each minimal vertex separator induces a subtree in a spanning tree $T$ for $G$ then $T$ is compatible with $G$.

Theorem

$G$ is dually chordal $\iff \exists T$ spanning tree such that every minimal vertex separator induces a subtree.
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**Theorem**

$G$ is dually chordal $\iff \exists$ $T$ spanning tree such that every minimal vertex separator induces a subtree.
**First conclusion:** Minimal vertex separators induce connected subgraphs.

**Property:** A family of subtrees of a tree is Helly.

**Second conclusion:** $S(G)$ is Helly.

**Property:** The intersection graph of a family of subtrees of a tree is chordal.

**Third conclusion:** $L(S(G))$ is chordal.

**Theorem**

$G$ is dually chordal

Each minimal separator induces a connected subgraph, $S(G)$ is Helly and $L(S(G))$ is chordal.
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**Theorem**

$G$ is dually chordal if and only if each minimal separator induces a connected subgraph, $S(G)$ is Helly, and $L(S(G))$ is chordal.
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**Theorem**

\[ G \text{ is dually chordal} \]

\[ \Updownarrow \]

Each minimal separator induces a connected subgraph, \( S(G) \) is Helly and \( L(S(G)) \) is chordal.
None of the three can be deduced from the others

\[ S(G) \text{ fails to be Helly} \]
\[ L(S(G)) \text{ fails to be chordal} \]
\[ \text{Some minimal separators are disconnected} \]
Idea of the proof

Property: If a family of sets is Helly and its intersection graph is chordal, then it can be represented as a family of subtrees of a tree.

Take $T$ with $V(T) = V(G)$ such that each minimal vertex separator of $G$ induces a subtree in $T$ and $p(T) := \sum_{uv \in E(T)} d(u, v)$ is minimum.

$T$ is a spanning tree and each minimal vertex separator of $G$ induces a subtree of $G$.

Conclusion: $G$ is dually chordal.
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Thank you!!