Bounds on the number of tessellations in graphs

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Outline

Section 1

Tessellations in Graphs

Definition

- Let $T_i = \{c_1, c_2, \cdots, c_n\}$ be a family of cliques of a graph G;
- T_i is a **tessellation** in G if and only if:
  - All cliques of T_i are **disjoint**, and;
  - The union of the cliques of T_i covers **all** vertices of G.
- Each clique in T_i is called a **cluster** (Portugal, 16).

Example

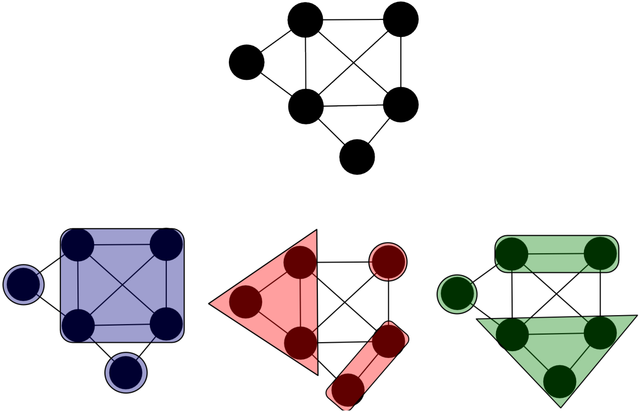
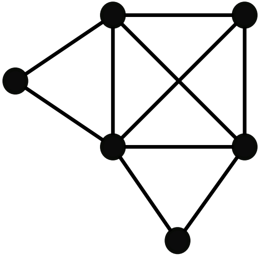


Figure 1: Tessellations of G. Each tessellation covers all vertices.

Number of Tessellations of a Graph

- A graph G is T-tessellable if T is the **smallest number** of tessellations such that the union of these tessellations covers all edges of G.



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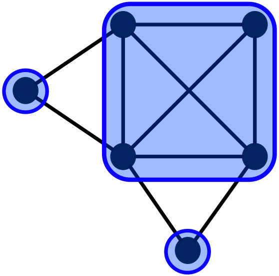

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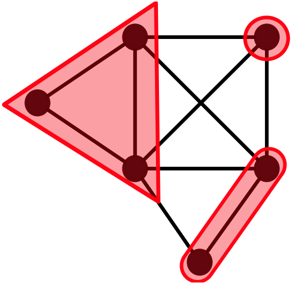

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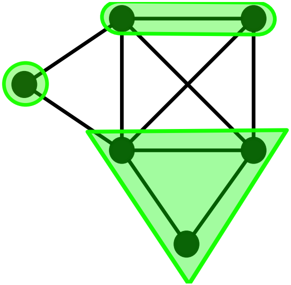

Motivation: Quantum walks

- Quantum walks are the model of a particle's tour through the vertices of a graph.
- In quantum walks there are quantum state representing the walker, and an evolution operator applied on the quantum state, moving the walker through the graph's vertices.
- Staggered quantum walks model uses tessellations in graphs to generate the evolution operators (Portugal et al., 16)

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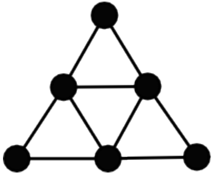
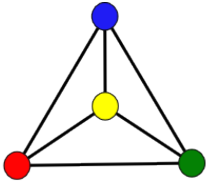
2-tessellable graphs

Proposition 1 ((PORTUGAL, 2016))

A graph is 2-tessellable if and only if its clique graph is 2-colorable.

- This proposition is not generalizable for $T > 2$.
  - The characterization of a T-tessellable graph is still an open problem.





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Section 2

Bounds on the number of tessellations

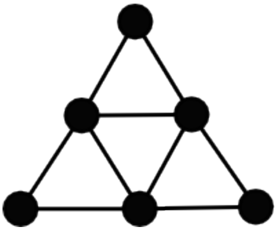
Upper Bound

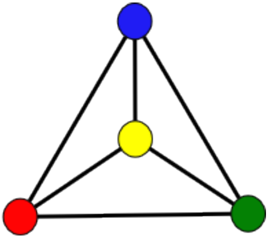
- We propose an upper bound for the number of tessellations.

Lemma 1

Given G and its clique graph $K(G)$, then $T(G) \leq \chi(K(G))$.

The proof's Idea of Lemma 1





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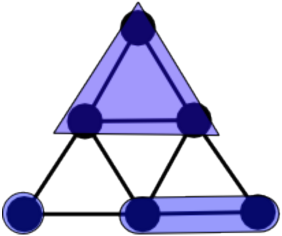

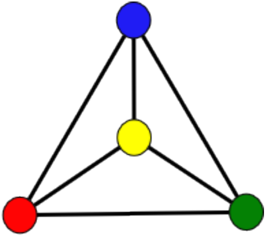

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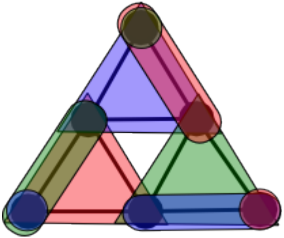

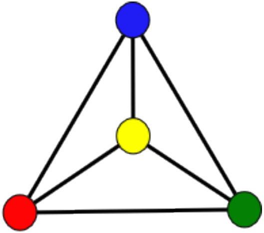

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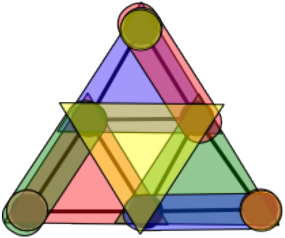
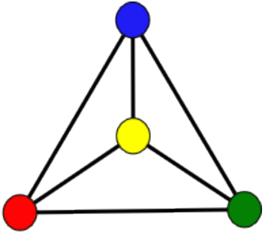


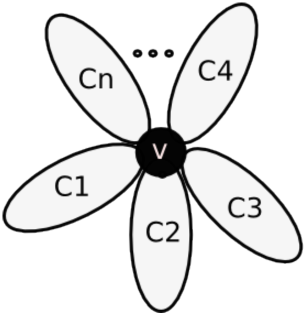


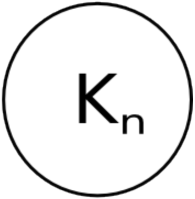
Upper bound is tight

Theorem 1

Let G be a graph s.t. there is a vertex $v \in V(G)$ which is a cut vertex and all cliques of G just share v. Then, $T(G) = \chi(K(G))$.

The proof's Idea of Theorem 1





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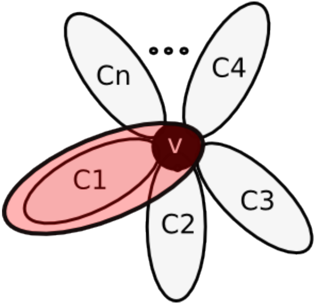

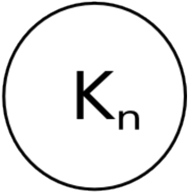

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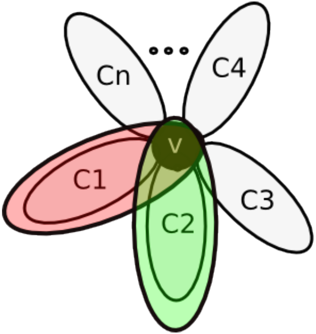

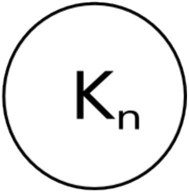

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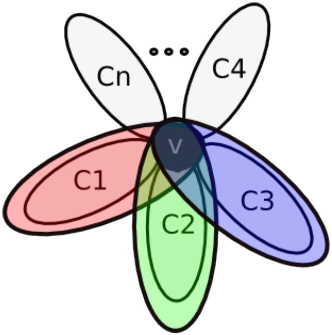

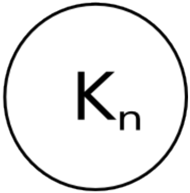

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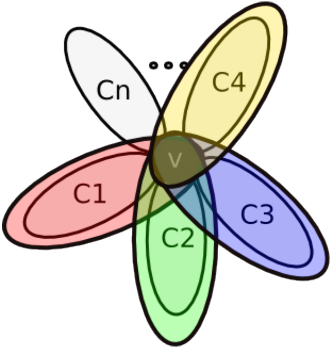

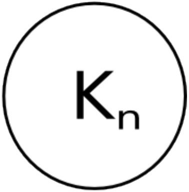

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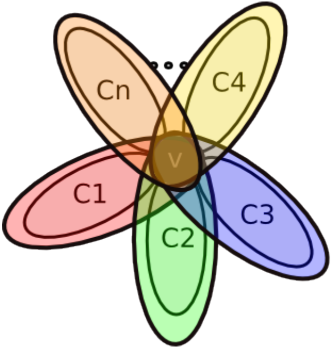

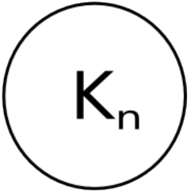

Lower Bound

- We propose a lower bound for the number of tessellations.

Lemma 2

Let G be a graph and $K(G)$ its clique graph. Then,
$T(G) \geq \lceil \frac{\chi(K(G))}{2} \rceil$.

The proof's Idea of Lemma 2

1. k-chromatic graph has at least k vertex with degree at least $k-1$.

2. For every graph G, $T(G) \leq \chi(K(G)) \leq \Delta(K(G)) + 1$.

The proof's Idea of Lemma 2

- Idea:
  - $m = m(K(G)) = \frac{\sum_i d_i}{2}$
  - By (1) we have that $m = \frac{\chi(\chi-1)}{2} + \frac{\sigma}{2}$, where σ is the remaining sum, and $0 \leq \sigma$.
  - $T(G) \leq \frac{\chi(\chi-1)}{2} + \frac{\sigma}{2}$.
  - So, $\frac{\chi^2}{2} - \frac{\chi}{2} + \frac{\sigma}{2} - T(G) \geq 0$.
  - Let us divide our problem in three cases, considering $\phi \geq 1$:
    - $(i) T(G) > \frac{\chi}{2}$, i.e., $T(G) = \frac{\chi}{2} + \phi$;
    - $(ii) T(G) = \frac{\chi}{2}$, and;
    - $(iii) T(G) < \frac{\chi}{2}$, i.e., $T(G) = \frac{\chi}{2} - \phi$

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The proof's Idea of Lemma 2

- In each of this three cases, we have to solve an inequation. By this inequation, it comes that to get an answer in $\mathbb{R}$ the discriminant Δ must be greater than or equal to zero. So:
  - $(i)\Delta = 1 - \sigma + 2\phi \geq 0 \rightarrow \sigma \leq 1 + 2\phi$;
  - $(ii)\Delta = 1 - \sigma \geq 0 \rightarrow \sigma \leq 1$, however;
  - $(iii)\Delta = 1 - \sigma - 2\phi \geq 0 \rightarrow \sigma \leq 1 - 2\phi < 0$, but $\sigma \geq 0$.
- Thus, we never have that $T(G) < \frac{\chi(K(G))}{2}$.

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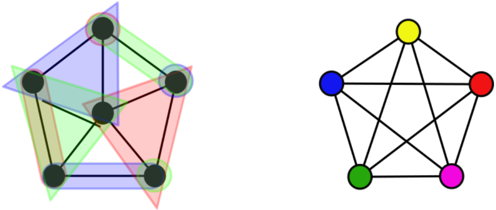
Lower bound is tight

- For a wheel graph w-wheel, s.t. $w > 4$, we have that $T(G) = \lceil \frac{\chi}{2} \rceil$.

Theorem 2

Let G be a w-wheel, s.t. $w > 4$. Then, $T(G) = \lceil \frac{\chi}{2} \rceil$.



Tight Bounds

Theorem 3

Let G be a graph and $K(G)$ its clique graph, and let $K(G)$ is non-bipartite. We have that $\lceil \frac{\chi(K(G))}{2} \rceil \leq T(G) \leq \chi(K(G))$.

Open Questions

- What is the number of tessellations for other classes?
- What is the complexity of this problem?

References

PORTUGAL, R. Staggered quantum walks on graphs. *arXiv preprint arXiv:1603.02210*, 2016.

PORTUGAL, R. et al. The staggered quantum walk model. *Quantum Information Processing*, Springer, v. 15, n. 1, p. 85–101, 2016.