On unit *d*–interval graphs

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Summary

Introduction
- Definitions
- Previous results
- Motivation

Unit d–interval graphs
- Characterization

Conclusions
- Current and future work

Definitions

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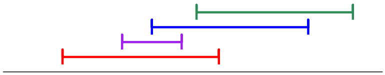

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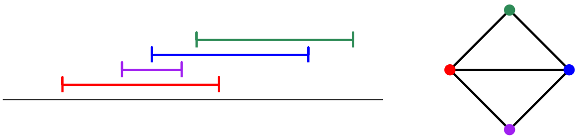

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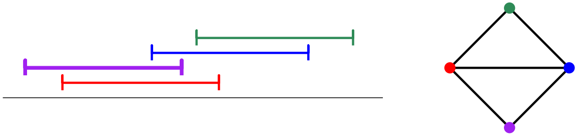

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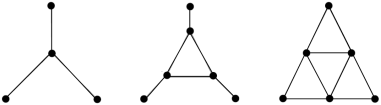
Theorem (Roberts 1969, Lekkerkerker-Boland 1962)

Given a graph G. The following are equivalent:

- G is unit interval graph,
- G is proper interval graph,
- G is {claw, net, tent}-free and chordal.



d–interval graphs

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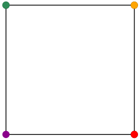

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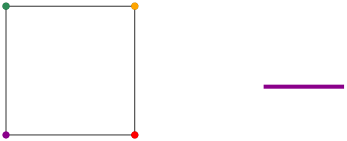

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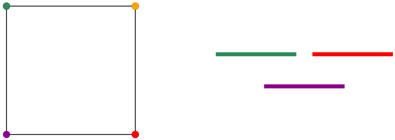

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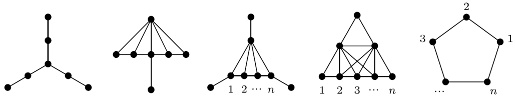
- Depth-two unit $\rightarrow$ P [$\mathcal{O}(n+m)$] (Jiang, 2013).

Motivation

- Lekkerkerker-Boland (1962) G is an interval graph $\Leftrightarrow$ G does not contain as induced subgraph any of these:



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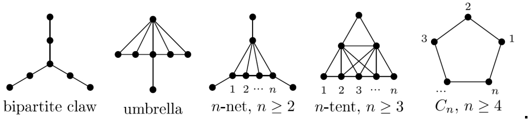

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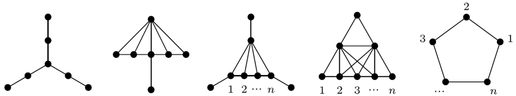

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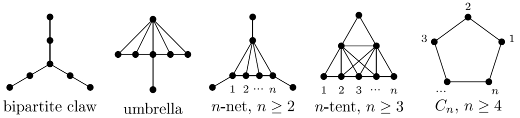

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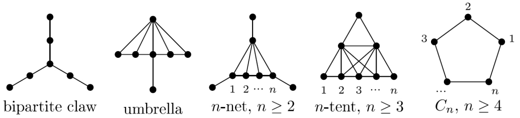

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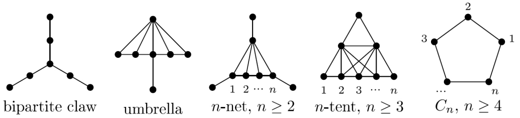

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  A REALLY SIMPLE CHARACTERIZATION!
- Encouraged by this, we study if it is possible to conclude to a similar simple characterization for unit d–interval graphs. That is, given an interval graph G, recognize if G is a unit d–interval graph or not.

Our results

Theorem

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For each interval $I \in \mathcal{M}$, we build a proper d–interval $J_1 \cup J_2 \cup \cdots \cup J_d$ as follows:

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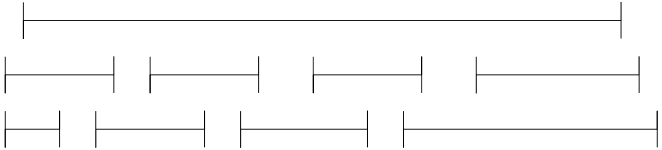

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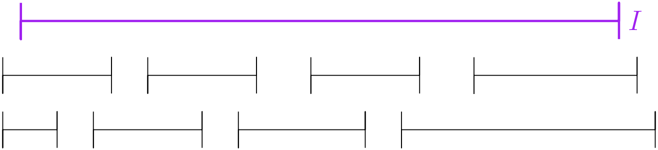

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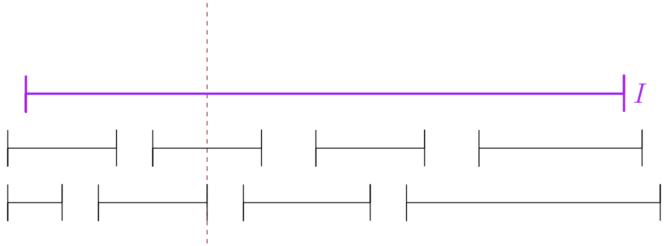

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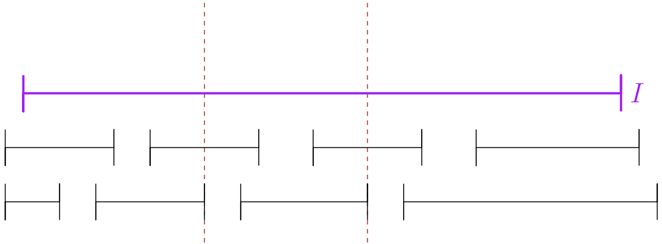

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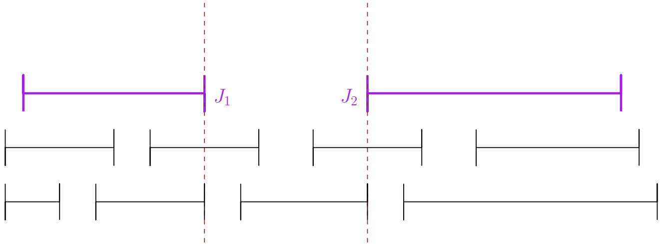

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Remark

The algorithm finds the minimum d for which an interval graph is a unit d–interval graph.

Conclusions

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- We found the minimum d for which an interval graph is a unit d–interval graph.

Current and future work

- As $\{$interval graphs$\} \subseteq \{$circular-arc graphs$\}$, extend the results for G circular-arc graphs. Find bounds on d such that G admits a unit d–interval model.

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- As {interval graphs} $\subseteq$ {circular-arc graphs}, extend the results for G circular-arc graphs. Find bounds on d such that G admits a unit d–interval model.

- Study the problem for unit d–track graphs, where each vertex is represented by one unit interval in d different real lines.

Thank you for your attention!

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F. S. Roberts, *Indifference graphs*, Proof Techniques in Graph Theory, Proceedings of the Second Ann Arbor Graph Theory Conference (New York), Academic Press, 1969, pp. 139–146.

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Complexities summary

	d–interval graphs	Paper
unrestricted	NP–complete ($d \geq 2$)	West and Shmoys [1984]
depth $\leq r$	NP–complete ($r \geq 3$)	West and Shmoys [1984]
balanced	NP–complete ($d = 2$)	Gambette and Vialette [2007]
unit	???	Gambette and Vialette [2007]
depth-two	???	Jiang [2013]
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Table : Current complexities of recognizing variants of d–interval graphs.