On the complexity of k-tuple total and total $\{k\}$-dominations

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joint work with G. Argiroffo[1] and P. Torres[1,2]

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VII LAWCG, La Plata, Argentina

8-11 Noviembre 2016

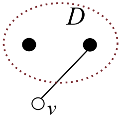
Introduction

G without isolated vertices, simple and finite graph

$V(G)$: vertex set of G

$N(v)$: (open) neighborhood of $v \in V(G)$

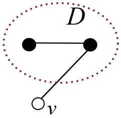
DOMINATION



$D \subseteq V(G)$ is *dominating set* if

$\forall v \in V(G) \setminus D,\ |N(v) \cap D| \geq 1$

TOTAL DOMINATION



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Main definitions: two variations of total domination

$f(D) := \sum_{v \in D} f(v)$ weight of $D \subseteq V(G)$

- $f : V(G) \mapsto \{0,1\}$ a *total dominating function* of G if

$$f(N(v)) \geq 1,\ \forall v \in V(G).$$

$\gamma_t(G)$: minimum value of $f(V(G))$ over all such f's

E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, *Total domination in graphs*, Networks (1980)

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- $f : V(G) \mapsto \{0, 1\}$ a *k-tuple total dominating function* of G if

$$f(N(v)) \geq k, \ \forall v \in V(G) \qquad (k \leq \delta(G)).$$

$\gamma_{\times k,t}(G)$: minimum value of $f(V(G))$ over all such f's

M. A. Henning and A. P. Kazemi, *k-tuple total domination in graphs*, Disc. App. Math. (2010).

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$\gamma_{\{k\},t} \leq \gamma_{\times k,t}(G)$ for $k \leq \delta(G)$, $\quad \gamma_{\times 1,t}(G) \leq \gamma_{\{k\},t} \leq k\gamma_{\times 1,t}(G)$

The problems

k-TUPLE TOTAL DOMINATION (k-DOM-T)

Instance: $G = (V(G), E(G))$, $j \in \mathbb{N}$
Question: Does G have a k-tuple total dominating function with weight at most j?

TOTAL $\{k\}$-DOMINATION ($\{k\}$-DOM-T)

Instance: $G = (V(G), E(G))$, $j \in \mathbb{N}$
Question: Does G have a total $\{k\}$-dominating function with weight at most j?

1-DOM-T=$\{1\}$-DOM-T= DOM-T

Known complexity results

"NP-c", "P" and " ? " mean NP-complete, polynomial and open problem, respectively.

Class	k-**DOM-T** (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-**DOM-T** (fixed $k \in \mathbb{Z}_+$)
Complete multipartite	P [1]	?
Chordal bipartite	P [2]	?
Doubly chordal	NP-c [2]	?
Bipartite	NP-c [2]	NP-c [3]
Split	NP-c [2]	?

[1] M. A. Henning and A. P. Kazemi, *k-tuple total domination in graphs*, Disc. App. Math. (2010).
[2] D. Pradhan, *Algorithmic aspects of k-tuple total domination in graphs*, Inf. Proc. Let. (2012).
[3] J. He and H. Liang, *Complexity of Total $\{k\}$-Domination and Related Problems*, LNCS (2011).

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Class	k-**DOM-T** (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-**DOM-T** (fixed $k \in \mathbb{Z}_+$)
Trees, Block graphs, Cacti	P [4]	?
Complete multipartite	P [1]	?
Chordal bipartite	P [2]	?
Doubly chordal	NP-c [2]	?
Bipartite	NP-c [2]	NP-c [3]
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[3] J. He and H. Liang, *Complexity of Total* $\{k\}$*-Domination and Related Problems*, LNCS (2011).
[4] Lan J., and G. J. Chang, *On the algorithmic complexity of k-tuple total domination*, Disc. App. Math. (2014).

Polynomial instances

k-DOM-T and $\{k\}$-DOM-T

Theorem

G, a graph; $k \in \mathbb{Z}_+$ and S_k, edgeless graph on k vertices. Then $\gamma_{\{k\},t}(G) = \gamma_{\times k,t}(G \bullet S_k)$.

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$G \bullet H$: lexicographic product

- $V(G \bullet H) = V(G) \times V(H)$
- (u_1, v_1) and (u_2, v_2) adjacent in $G \bullet H$ when
  - ▷ u_1 adjacent to u_2 in G
  - ▷ $u_1 = u_2$ and v_1 adjacent to v_2 in H.

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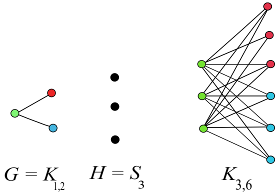

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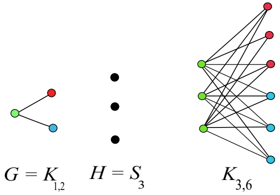
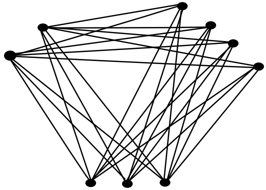


Corollary

*k fixed and $\mathcal{F}$, $\mathcal{S}$ s.t. if $G \in \mathcal{F}$ then $G \bullet S_k \in \mathcal{S}$.
If k-DOM-T is polynomial time solvable on $\mathcal{S}$, then $\{k\}$-DOM-T on $\mathcal{F}$ also is.*

$\{k\}$-DOM-T on complete multipartite graphs

G is *complete multipartite* if $V(G)$ can be partitioned into non-empty independent sets with all possible edges between any of these sets.



$\{k\}$-DOM-T on complete multipartite graphs

$p \geq 2, \quad n_1 \leq n_2 \leq \cdots \leq n_p, \qquad G = K_{n_1,\ldots,n_p}$ complete p-partite graph. Then,

- $p \geq k+1$, then $\gamma_{\times k,t}(G) = k+1$;
- $p = k$ and $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k,t}(G) = k+2$;
- $p \leq k-1$ and $n_1 \geq \lceil \frac{k}{p-1} \rceil$, then $\gamma_{\times k,t}(G) = \lceil \frac{pk}{p-1} \rceil$ [1].

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Since $G = K_{n_1,\ldots,n_p} \Rightarrow G \bullet S_k = K_{kn_1,\ldots,kn_p}$.

Corollary

$p \geq 2$, $G = K_{n_1,\ldots,n_p}$ $(n_1 \leq \cdots \leq n_p)$ complete p-partite graph:

1. $\gamma_{\{k\},t}(G) = k+1$ *if* $p \geq k+1$,
2. $\gamma_{\{k\},t}(G) = k+2$ *if* $p = k$,
3. $\gamma_{\{k\},t}(G) = \lceil \frac{pk}{p-1} \rceil$ *if* $p \leq k-1$.

$\{k\}$-DOM-T on Chordal bipartite graphs

A bipartite graph is *chordal bipartite* if each cycle of length at least 6 has a chord.

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1. If G is $\{hole, \ triangle\}-$ free then $G \bullet S_k$ also is.
2. k-DOM-T is polynomial on chordal bipartite graphs ([2]).

Corollary

$\{k\}$-DOM-T is polynomial time solvable on chordal bipartite graphs.

[2] D. Pradhan, *Algorithm aspects of k-tuple total domination in graphs*, Inf. Proc. Let. (2012).

Other graph classes studied

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- Since we could expressed k-DOM-T in LinEMSOL, k-DOM-T is polynomial in bounded *clique-width* graphs.
  Besides, if $cwd(G)$ is bounded then also is $cwd(G \bullet S_k)$. Then, $\{k\}$-DOM-T is polynomial in bounded *clique-width* graphs.

k-DOM-T in LinEMSOL

$X(u) \quad \leftrightarrow \quad u \in X.$

Finding $\gamma_{\times k,t}(G) \Leftrightarrow$ finding z s.t.

$|z(X)|_1 = \text{mín}\{|z'(X)|_1 : \theta(X)$ is true for G and $z'\}$, where

$$\theta(X) = \ \forall v \ \left(\bigwedge_{1 \leq r \leq k} A_r(X, v, u_1, \ldots, u_r) \right)$$

$A_1(X, v, u_1) := \exists u_1 \, [X(u_1) \wedge adj(v, u_1)],$

and for $r > 1$:

$$A_r(X, v, u_1, \ldots, u_r) := \exists u_r \left[X(u_r) \wedge adj(v, u_r) \wedge \bigwedge_{1 \leq i \leq r-1} \neg(u_r = u_i) \right].$$

NP-completeness results

Bipartite planar graphs

1-DOM-T on bipartite planar graphs

Theorem

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VERTEX COVER PROBLEM
Instance: $G = (V(G), E(G)), \quad j \in \mathbb{N}$.
Question: : $\exists\ C \subseteq V(G), \quad |C| \leq j$ s.t. every xy has either $x \in C$ or $y \in C$?

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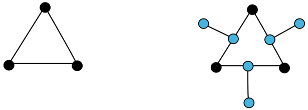
A reduction from VC on planar graphs to DOM-T:



1-DOM-T on bipartite planar graphs

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VERTEX COVER PROBLEM (VC)
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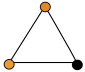
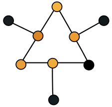


a vertex cover



a total dom set

2,3-DOM-T on bipartite planar graphs

Theorem

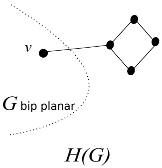
For $k \in \{2,3\}$, $\gamma_{\times k,t}(H(G)) = \gamma_{\times (k-1),t}(G) + 2^k|V(G)|$.

2,3-DOM-T on bipartite planar graphs

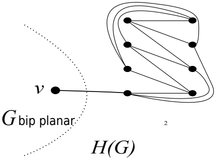
Theorem

For $k \in \{2, 3\}$, $\gamma_{\times k,t}(H(G)) = \gamma_{\times (k-1),t}(G) + 2^k |V(G)|$.

$k = 2$



$k = 3$



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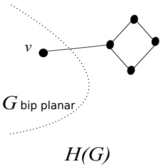

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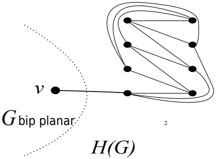


Corollary

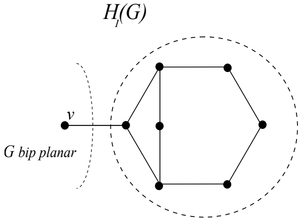
For $k \in \{2,3\}$, k-DOM-T is NP-c on bipartite planar graphs.

Remark: For a bipartite planar graph and $k \geq 4$, there is no k-tuple total dominating function.

$\{k\}$-DOM-T on bipartite planar graphs

Theorem

For $k \in \mathbb{Z}_+$, $\gamma_{\{k\},t}(H_1(G)) = \gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) + |V(G)|\gamma_{\{k\},t}(C_6)$.



Lemma

C_n chordless cycle with $n \geq 3$. Then,

- $\gamma_{\{k\},t}(C_n) = \lceil \frac{nk}{2} \rceil + 1$ *if k is odd and $n = 2(mod\ 4)$,*
- $\gamma_{\{k\},t}(C_n) = \lceil \frac{nk}{2} \rceil$ *in other case.*

$\gamma_{\{k\},t}(C_6) = 3k + 1$ if k odd; $\gamma_{\{k\},t}(C_6) = 3k$ if k even.

$\{k\}$-DOM-T on bipartite planar graphs

Theorem

For $k \in \mathbb{Z}_+$, $\gamma_{\{k\},t}(H_1(G)) = \gamma_{\{\lfloor \frac{k}{2} \rfloor\},t}(G) + |V(G)|\gamma_{\{k\},t}(C_6)$.

Corollary

For every fixed $k \in \mathbb{Z}_+$, *$\{k\}$-DOM-T is NP-c on bipartite planar graphs.*

Partial summary

Class	k-DOM-T (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-DOM-T (fixed $k \in \mathbb{Z}_+$)
Bounded clique-width	P	P
Complete multipartite	P	P
Chordal bipartite	P	P
Doubly chordal	NP-c	?
Bipartite	NP-c	NP-c
Bipartite planar	NP-c (for $k = 2, 3$)	NP-c
Planar	NP-c (for $k = 2, 3$)	NP-c

Further study: NP-completeness of $4, 5$-DOM-T on planar graphs

Remark: For a planar graph and $k \geq 6$, there is no k-tuple total dom function.

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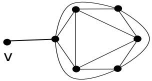
Theorem

$$\gamma_{\times 4,t}(P(G)) = \gamma_{\times 3,t}(G) + 6|V(G)|.$$

$k = 4$



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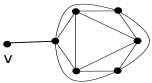

Theorem

$$\gamma_{\times 5,t}(P(G)) = \gamma_{\times 4,t}(G) + 12|V(G)|.$$

Corollary

For $k \in \{4,5\}$, k-DOM-T is NP-c on planar graphs.

Final summary

Class	k-**DOM-T** (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-**DOM-T** (fixed $k \in \mathbb{Z}_+$)
Bounded clique-width	P	P
Complete multipartite	P	P
Chordal bipartite	P	P
Doubly chordal	NP-c	**?**
Bipartite	NP-c	NP-c
Bipartite planar	NP-c (for $k = 2, 3$)	NP-c
Planar	NP-c (for $k = 2, 3, 4, 5$)	NP-c

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Bounded clique-width	P	P
Complete multipartite	P	P
Chordal bipartite	P	P
Doubly chordal	NP-c	**?**
Bipartite	NP-c	NP-c
Bipartite planar	NP-c (for $k = 2, 3$)	NP-c
Planar	NP-c (for $k = 2, 3, 4, 5$)	NP-c
Split	NP-c	NP-c

Some open problems

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   Find an algorithm for any of the problems for chordal bipartite graphs.

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4. ......

THANK YOU! / ¡GRACIAS! / ¡OBRIGADO!