On the complexity of $k$-tuple total and total $\{k\}$-dominations

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joint work with G. Argiroffo$^1$ and P. Torres$^{1,2}$

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VII LAWCG, La Plata, Argentina
8-11 Noviembre 2016
$G$ without isolated vertices, simple and finite graph

$V(G)$: vertex set of $G$

$N(v)$: (open) neighborhood of $v \in V(G)$

**DOMINATION**

$D \subseteq V(G)$ is dominating set if

$\forall v \in V(G) \setminus D$, $|N(v) \cap D| \geq 1$

**TOTAL DOMINATION**

$D \subseteq V(G)$ is total dominating set if

$\forall v \in V(G)$, $|N(v) \cap D| \geq 1$
Main definitions: two variations of total domination

\[ f(D) := \sum_{v \in D} f(v) \text{ weight of } D \subseteq V(G) \]

- \( f : V(G) \rightarrow \{0, 1\} \) a total dominating function of \( G \) if
  \[ f(N(v)) \geq 1, \forall v \in V(G). \]

\( \gamma_t(G) \): minimum value of \( f(V(G)) \) over all such \( f \)’s

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  \[ f(N(v)) \geq k, \ \forall v \in V(G) \quad (k \leq \delta(G)). \]

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\[ \gamma\{k\}, t \leq \gamma \times k, t(G) \text{ for } k \leq \delta(G), \]
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\( \gamma_{\{k\},t}(G) \): minimum value of \( f(V(G)) \) over all such \( f \)'s


\[ \gamma_{\{k\},t} \leq \gamma_{\times k,t}(G) \text{ for } k \leq \delta(G), \quad \gamma_{\times 1,t}(G) \leq \gamma_{\{k\},t} \leq k \gamma_{\times 1,t}(G) \]
The problems

**k-TUPLE TOTAL DOMINATION** (k-DOM-T)

*Instance:* \( G = (V(G), E(G)), j \in \mathbb{N} \)

*Question:* Does \( G \) have a \( k \)-tuple total dominating function with weight at most \( j \)?

**TOTAL \( \{k\}\)-DOMINATION** (\( \{k\}\)-DOM-T)

*Instance:* \( G = (V(G), E(G)), j \in \mathbb{N} \)

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1-DOM-T=\{1\}-DOM-T= DOM-T
Known complexity results

“NP-c”, “P” and “?” mean NP-complete, polynomial and open problem, respectively.

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<thead>
<tr>
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Polynomial instances
Theorem

$G$, a graph; $k \in \mathbb{Z}_+$ and $S_k$, edgeless graph on $k$ vertices. Then $\gamma_{\{k\},t}(G) = \gamma_{\times k,t}(G \cdot S_k)$.
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$G$, a graph; $k \in \mathbb{Z}_+$ and $S_k$, edgeless graph on $k$ vertices. Then $\gamma_{\{k\},t}(G) = \gamma_{\times k,t}(G \bullet S_k)$.

$G \bullet H$: lexicographic product

- $V(G \bullet H) = V(G) \times V(H)$
- $(u_1, v_1)$ and $(u_2, v_2)$ adjacent in $G \bullet H$ when
  - $u_1$ adjacent to $u_2$ in $G$
  - $u_1 = u_2$ and $v_1$ adjacent to $v_2$ in $H$. 
**Theorem**

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$G = K_{1,2}$, $H = S_3$, $K_{3,6}$
Theorem

Let $G$, a graph; $k \in \mathbb{Z}_+$ and $S_k$, edgeless graph on $k$ vertices. Then $\gamma_{\{k\},t}(G) = \gamma_{\times k,t}(G \bullet S_k)$.

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Corollary

$k$ fixed and $F, S$ s.t. if $G \in F$ then $G \bullet S_k \in S$. If $k$-DOM-T is polynomial time solvable on $S$, then $\{k\}$-DOM-T on $F$ also is.
$G$ is *complete multipartite* if $V(G)$ can be partitioned into non-empty independent sets with all possible edges between any of these sets.

\[ \{k\}\text{-DOM-T on complete multipartite graphs} \]

\[ G \text{ is complete multipartite if } V(G) \text{ can be partitioned into non-empty independent sets with all possible edges between any of these sets.} \]

\[ G = K_{n_1}, \ldots, K_{n_p} \text{ complete } p\text{-partite graph. Then, } p \geq k + 1, \text{ then } \gamma \times k, t(G) = k + 1; \]

\[ p = k \text{ and } k - 1 \sum_{i=1}^{p} n_i \geq k, \text{ then } \gamma \times k, t(G) = k + 2; \]

\[ p \leq k - 1 \text{ and } n_1 \geq \lceil kp - 1 \rceil, \text{ then } \gamma \times k, t(G) = \lceil pk - 1 \rceil \]

$p \geq 2, \ n_1 \leq n_2 \leq \cdots \leq n_p, \ G = K_{n_1, \ldots, n_p}$ complete $p$-partite graph. Then,

- $p \geq k + 1$, then $\gamma_{\times k, t}(G) = k + 1$;
- $p = k$ and $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k, t}(G) = k + 2$;
- $p \leq k - 1$ and $n_1 \geq \left\lceil \frac{k}{p-1} \right\rceil$, then $\gamma_{\times k, t}(G) = \left\lceil \frac{pk}{p-1} \right\rceil$ [1].

$\{k\}$-DOM-T on complete multipartite graphs

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- $p \geq k + 1$, then $\gamma_{\times k, t}(G) = k + 1$;
- $p = k$ and $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k, t}(G) = k + 2$;
- $p \leq k - 1$ and $n_1 \geq \lceil \frac{k}{p-1} \rceil$, then $\gamma_{\times k, t}(G) = \lceil \frac{pk}{p-1} \rceil$ \[1\].


Since $G = K_{n_1, \ldots, n_p} \Rightarrow G \bullet S_k = K_{kn_1, \ldots, kn_p}$.

**Corollary**

$p \geq 2, \ G = K_{n_1, \ldots, n_p} \ (n_1 \leq \cdots \leq n_p)$ complete $p$-partite graph:

1. $\gamma_{\{k\}, t}(G) = k + 1$ if $p \geq k + 1$,
2. $\gamma_{\{k\}, t}(G) = k + 2$ if $p = k$,
3. $\gamma_{\{k\}, t}(G) = \lceil \frac{pk}{p-1} \rceil$ if $p \leq k - 1$. 
A bipartite graph is *chordal bipartite* if each cycle of length at least 6 has a chord.
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A graph is *chordal bipartite* if it is \{hole, triangle\}− free.

\[ hole \equiv \text{cycle with at least 5 vertices}. \]
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\( hole \equiv \) cycle with at least 5 vertices.

1. If \( G \) is \{hole, triangle\}− free then \( G \cdot S_k \) also is.
2. \( k\)-DOM-T is polynomial on chordal bipartite graphs ([2]).

**Corollary**

\( \{k\}\)-DOM-T is polynomial time solvable on chordal bipartite graphs.

Other graph classes studied

- \( \mathcal{F} \)-free graphs, for a family \( \mathcal{F} \) with certain structure.
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- $\mathcal{F}$-free graphs, for a family $\mathcal{F}$ with certain structure.

Since we could express $k$-DOM-T in LinEMSOL, $k$-DOM-T is polynomial in bounded \textit{clique-width} graphs. Besides, if $cwd(G)$ is bounded then also is $cwd(G \bullet S_k)$. Then, \{k\}-DOM-T is polynomial in bounded \textit{clique-width} graphs.
$X(u) \leftrightarrow u \in X$. 

Finding $\gamma_{\times k,t}(G) \Leftrightarrow$ finding $z$ s.t. 

$|z(X)|_1 = \min\{|z'(X)|_1 : \theta(X) \text{ is true for } G \text{ and } z'\}$, where 

$$
\theta(X) = \forall v \left( \bigwedge_{1 \leq r \leq k} A_r(X, v, u_1, \ldots, u_r) \right)
$$

$A_1(X, v, u_1) := \exists u_1 [X(u_1) \land \text{adj}(v, u_1)]$, 

and for $r > 1$:

$$
A_r(X, v, u_1, \ldots, u_r) := \exists u_r \left[ X(u_r) \land \text{adj}(v, u_r) \land \bigwedge_{1 \leq i \leq r-1} \neg(u_r = u_i) \right].
$$
NP-completeness results
Bipartite planar graphs

V. Leoni

$k$-tuple total & total \( \{k\} \)-dominations
1-DOM-T on bipartite planar graphs

Theorem

1-DOM-T is NP-complete on bipartite planar graphs.

Vertex cover in $G$: some vertices intersecting all the edges

VERTEX COVER PROBLEM

Instance: $G = (V(G), E(G))$, $j \in \mathbb{N}$.

Question: $\exists C \subseteq V(G)$, $|C| \leq j$ s.t. every $xy$ has either $x \in C$ or $y \in C$?

A reduction from VC on planar graphs to DOM-T:
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![Graphs](image-url)
Theorem

**DOM-T is NP-complete on bipartite planar graphs.**

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**VERTEX COVER PROBLEM (VC)**

**Instance:** \( G = (V(G), E(G)), \ j \in \mathbb{N}. \)

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A reduction from VC on planar graphs to DOM-T:
Theorem

For $k \in \{2, 3\}$, $\gamma_{x,k,t}(H(G)) = \gamma_{x,(k-1),t}(G) + 2^k |V(G)|$. 

Corollary

For $k \in \{2, 3\}$, $k$-DOM-T is NP-c on bipartite planar graphs.

Remark: For a bipartite planar graph and $k \geq 4$, there is no $k$-tuple total domination.
Theorem

For $k \in \{2, 3\}$, $\gamma_{x,k,t}(H(G)) = \gamma_{x,(k-1),t}(G) + 2^k |V(G)|$.
2,3-DOM-T on bipartite planar graphs

**Theorem**

For $k \in \{2, 3\}$, $\gamma_{\times k,t}(H(G)) = \gamma_{\times (k-1),t}(G) + 2^k|V(G)|$.

**Corollary**

For $k \in \{2, 3\}$, $k$-DOM-T is NP-c on bipartite planar graphs.

**Remark**: For a bipartite planar graph and $k \geq 4$, there is no $k$-tuple total dominating function.
Theorem

For $k \in \mathbb{Z}_+$, $\gamma\{k\}, t(H_1(G)) = \gamma\left\lfloor \frac{k}{2} \right\rfloor, t(G) + |V(G)| \gamma\{k\}, t(C_6)$.

Lemma

$C_n$ chordless cycle with $n \geq 3$. Then,

- $\gamma\{k\}, t(C_n) = \left\lfloor \frac{nk}{2} \right\rfloor + 1$ if $k$ is odd and $n = 2(\text{mod } 4)$,
- $\gamma\{k\}, t(C_n) = \left\lfloor \frac{nk}{2} \right\rfloor$ in other case.

$\gamma\{k\}, t(C_6) = 3k + 1$ if $k$ odd; $\gamma\{k\}, t(C_6) = 3k$ if $k$ even.
Theorem

For \( k \in \mathbb{Z}_+ \), \( \gamma_{\{k\},t}(H_1(G)) = \gamma_{\lfloor \frac{k}{2} \rfloor},t(G) + |V(G)|\gamma_{\{k\},t}(C_6) \).

Corollary

For every fixed \( k \in \mathbb{Z}_+ \), \( \{k\}\)-DOM-T is NP-c on bipartite planar graphs.
### Partial summary

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Further study: NP-completeness of 4, 5-DOM-T on planar graphs

**Remark**: For a planar graph and $k \geq 6$, there is no $k$-tuple total dom function.
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**Theorem**

$$\gamma_{\times 4,t}(P(G)) = \gamma_{\times 3,t}(G) + 6|V(G)|.$$
Further study: NP-completeness of 4, 5-DOM-T on planar graphs

**Remark:** For a planar graph and \( k \geq 6 \), there is no \( k \)-tuple total dom function.

**Theorem**

\[
\gamma_{4,t}(P(G)) = \gamma_{3,t}(G) + 6|V(G)|. 
\]

**Theorem**

\[
\gamma_{5,t}(P(G)) = \gamma_{4,t}(G) + 12|V(G)|. 
\]

**Corollary**

For \( k \in \{4, 5\} \), \( k \)-DOM-T is NP-c on planar graphs.
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Some open problems

1. Solve the c.c. of \( \{k\}\text{-DOM-T} \) for doubly chordal graphs.
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2. Characterize graphs that satisfy $\gamma_{\{k\},t}(G) = \gamma_{\times k,t}(G)$.
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1. Solve the c.c. of \( \{k\}\text{-DOM-T} \) for doubly chordal graphs.

2. Characterize graphs that satisfy \( \gamma\{k\},t(G) = \gamma_{k,t}(G) \).

3. Chordal bipartite graphs are \( \mathcal{F}\text{-free graphs} \) \((\mathcal{F} = \{\text{hole, triangle}\})\).
   Find an algorithm for any of the problems for chordal bipartite graphs.
1. Solve the c.c. of $\{k\}$-DOM-T for doubly chordal graphs.

2. Characterize graphs that satisfy $\gamma_{k,t}(G) = \gamma_{k,\times k}(G)$.

3. Chordal bipartite graphs are $\mathcal{F}$-free graphs ($\mathcal{F} = \{\text{hole, triangle}\}$). Find an algorithm for any of the problems for chordal bipartite graphs.

4. ......

THANK YOU! / ¡GRACIAS! / ¡OBRIGADO!