Total dominating sequences in trees, split graphs, and under modular decomposition

Boštjan Brešar   Tim Kos

Graciela Nasini   Pablo Torres

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- Total dominating set: $D \subseteq V(G)$ for all $v \in V(G)$, $D \cap N(v) \neq \emptyset$.

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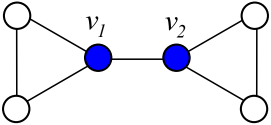
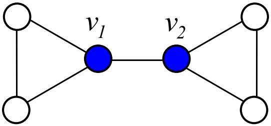
- Grundy total domination number: $\gamma_{\rm gr}^t(G)$: maximum length of a total dominating sequence of G.

Introduction



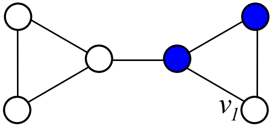
Total dominating set

INTRODUCTION



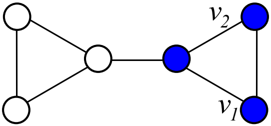
Total dominating sequence

INTRODUCTION



Legal sequence

INTRODUCTION



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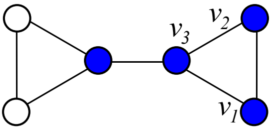

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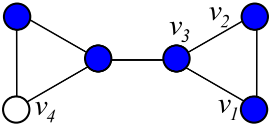

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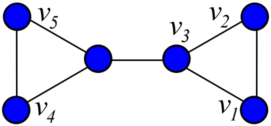
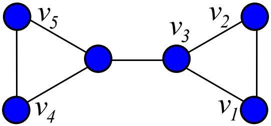


Grundy total dominating sequence (Gtds)

Introduction



Grundy total dominating sequence (Gtds)

$$\gamma_{\rm gr}^t(G) = 5.$$

INTRODUCTION

- n even, $\gamma^t_{\rm gr}(P_n) = n$.

  $n \geq 3$ odd, $\gamma^t_{\rm gr}(P_n) = n-1$.
- $n \geq 4$ even, $\gamma^t_{\rm gr}(C_n) = n-2$.

  $n \geq 3$ odd, $\gamma^t_{\rm gr}(C_n) = n-1$.
- $\gamma_t(G) \leq \gamma^t_{\rm gr}(G) \leq n - \delta(G) + 1$.
- If T is a tree, $\gamma^t_{\rm gr}(T) = n$ iff T has a perfect matching.
- $\frac{n}{\Delta} \leq \gamma^t_{\rm gr}(G)$.
- If G is connected and $\frac{n}{\Delta} = \gamma^t_{\rm gr}(G)$ then $G = K_{\Delta,\Delta}$.
- $\gamma^t_{\rm gr}(G) = 2$ iff G is a complete multipartite graph.

B. Brešar, M. A. Henning, and D. F. Rall, Total dominating sequences in graphs, *Discrete Math.* **339** (2016), 1165–1676

γ_{gr}^{t} OF INDUCED SUBGRAPHS

- $v \in V(G)$, $L(v) = \{\ell \in N(v) : deg(\ell) = 1\} \neq \emptyset$.
- S' a Gtds of $G' = G \setminus (\{v\} \cup L(v))$

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Then $S = (\ell) \oplus S' \oplus (v)$ is a Gtds of G and $\gamma^t_{gr}(G) = \gamma^t_{gr}(G') + 2$.

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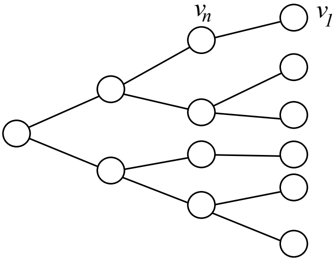

$(v_1) \oplus S' \oplus (v_n)$

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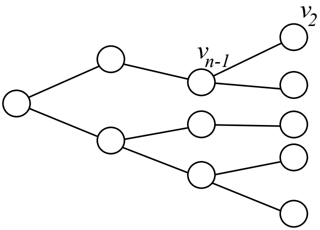

$(v_1, v_2) \oplus S' \oplus (v_{n-1}, v_n)$

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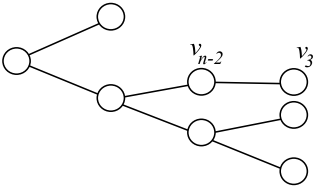

$(v_1, v_2, v_3) \oplus S' \oplus (v_{n-2}, v_{n-1}, v_n)$

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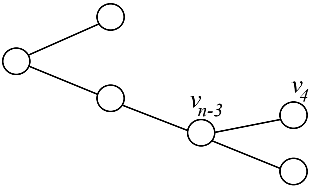

$$(v_1, v_2, v_3, v_4) \oplus S' \oplus (v_{n-3}, v_{n-2}, v_{n-1}, v_n)$$

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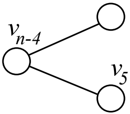

$$(v_1, v_2, v_3, v_4, v_5, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n)$$

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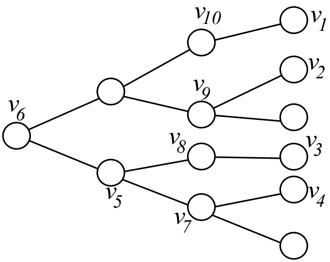

$(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$

$\gamma^t_{\rm gr}$ OF INDUCED SUBGRAPHS

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Algorithm 1: GrundyForest(T, S)

Input: A forest T.
Output: A Grundy total dominating sequence S of T.

```
if E(T) = ∅ then
    S = (),
    STOP.
Choose ℓ ∈ L(T) and let T' = T \ (s(ℓ) ∪ L(s(ℓ)))
Execute GrundyForest(T',S')
S = (ℓ) ⊕ S' ⊕ (s(ℓ))
STOP.
```

γ_{gr}^t OF INDUCED SUBGRAPHS

Lemma

If G is a graph, $v \in V(G)$ and $G' = G - v$, then

$$\gamma_{gr}^t(G) - 2 \leq \gamma_{gr}^t(G') \leq \gamma_{gr}^t(G).$$

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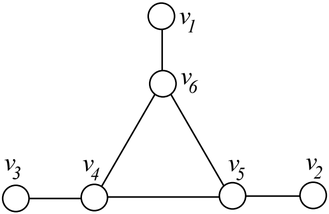

Net graph G

$\gamma^t_{\rm gr}$ OF INDUCED SUBGRAPHS

LEMMA

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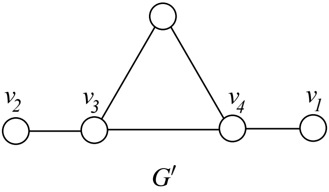

γ_{gr}^t OF INDUCED SUBGRAPHS

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γ^t_{gr} OF INDUCED SUBGRAPHS

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Then, S' is a Gtds in G and $\gamma^t_{gr}(G) = \gamma^t_{gr}(G - v')$.

Distance-hereditary graphs

A graph G is distance-hereditary (DH) if for each induced connected subgraph G' of G and all $x, y \in V(G')$, $d_{G'}(x,y) = d_G(x,y)$.

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G is DH if and only if it can be constructed from K_1 by a sequence of three operation: adding a pendant vertex, creating a true twin vertex ($N[v'] = N[v]$) and creating a false twin vertex (Bandelt, Mulder, 1986).

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A pruning sequence of G is a total ordering $\sigma = [x_1, \ldots, x_{|V(G)|}]$ of $V(G)$ and a sequence Q of triples $q_i = (x_i, Z, y_i)$ for $i = 1, \ldots, |V(G)| - 1$, where $Z \in \{P, F, T\}$ and such that, for $i \in \{1, \ldots, |V(G)| - 1\}$, if $G_i = G \setminus \{x_1, \ldots, x_{i-1}\}$ then,

- $Z = P$, if x_i is a leaf and $y_i = s(x_i)$ in G_i,
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1. DH graphs are characterized as the graphs which admit a pruning sequence (Hammer, Maffray, 1990).
2. DH graphs can be recognized in $O(|V(G)| + |E(G)|)$ (Damiand, Habib, Paul, 2001).
3. Given a DH, a pruning sequence can be computed in $O(|V(G)| + |E(G)|)$ (Damiand, Habib, Paul, 2001).

Bipartite distance-hereditary graphs

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Algorithm 4: GrundyBDH(G, S)

Input: A bipartite distance-hereditary graph G.
Output: A Grundy total dominating sequence S of G.

if $E(G) = \emptyset$ **then**
  $S = ()$
  STOP.
Obtain a pruning sequence $Q = [q_1, \ldots, q_{|V(G)|-1}]$ of G
for $i = 1$ *to* $|V(G)| - 1$ **do**
  if $q_i \neq (x_i, F, y_i)$ **then**
    $G' = G \setminus \{x_1, \ldots, x_i, y_i\}$
    Execute GrundyBDH(G',S')
    $S = (x_i) \oplus S' \oplus (y_i)$
    STOP.

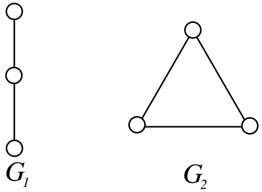
Bipartite distance-hereditary graphs

Theorem

Algorithm GrundyBDH returns a Grundy total dominating sequence of an arbitrary bipartite distance-hereditary graph G. The complexity of Algorithm GrundyBDH is $O(|V(G)|(|V(G)|+|E(G)|))$.

DISJOINT UNION AND JOIN

- G no connected, $G = G_1 + G_2$
  $V(G) = V(G_1) \cup V(G_2)$; $E(G) = E(G_1) \cup E(G_2)$.



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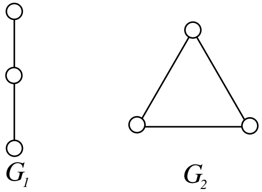

- $\overline{G}$ no connected, $G = G_1 \vee G_2$
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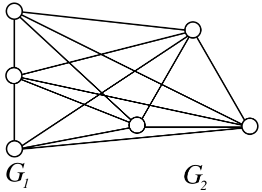

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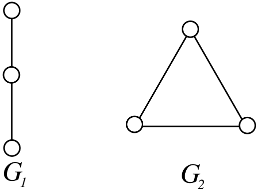

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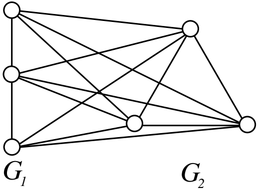

  - G connected and $\overline{G}$ connected $\rightarrow$ G modular graph.

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- G no connected, $G = G_1 + G_2$
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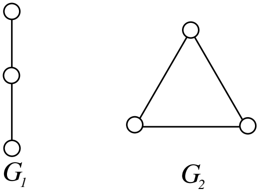
  $S_1 \oplus S_2$ is a Gtds of $G_1 + G_2$,
  $\gamma^t_{gr}(G_1 + G_2) = \gamma^t_{gr}(G_1) + \gamma^t_{gr}(G_2)$.



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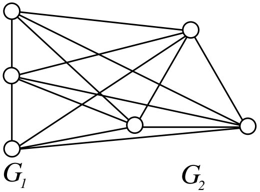

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Disjoint union and join

Lemma

Let S_1 *and* S_2 *be, respectively, Gtds of* G_1 *and* G_2*. Let* $G = G_1 \vee G_2$*.*

1. *If* $\eta(G_1) = \eta(G_2) = 0$ *and* $|\widehat{S}_1| \geq |\widehat{S}_2|$*,* S_1 *is a Gtds of* G*.*
2. *If* $\eta(G_1) = \eta(G_2) = 1$*,* $|\widehat{S}_1| \geq |\widehat{S}_2|$ *and* v *is an isolated vertex of* G_1*, then* $S = (v) \oplus S_1 \oplus (w)$ *is a Gtds of* G*, for any* $w \in V(G_2)$*.*
3. *If* $\eta(G_1) = 1$*,* $\eta(G_2) = 0$*,* $|\widehat{S}_1| + 2 \geq |\widehat{S}_2|$ *(resp.* $|\widehat{S}_1| + 2 \leq |\widehat{S}_2| - 1$*) then* $S = (v) \oplus S_1 \oplus (w)$*, for any isolated vertex* v *of* G_1 *and any* $w \in V(G_2)$*, (resp.* $S = S_2$*) is a Gtds of* G*.*

Hence,

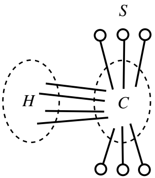
$$\gamma^t_{\mathrm{gr}}(G) = max\{\gamma^t_{\mathrm{gr}}(G_1) + 2\eta(G_1), \gamma^t_{\mathrm{gr}}(G_2) + 2\eta(G_2)\}.$$

$\eta(G)$ has value one if the graph has isolated vertices and zero, otherwise.

Spider graphs

Spider graph: G such that $V(G)$ can be partitioned into a stable set S, a clique C and a head H:

- $S = \{s_1, \ldots, s_r\}$, $C = \{c_1, \ldots, c_r\}$, $r \geq 2$, r: weight of G
- $N(S) \cap H = \emptyset$, $\forall s \in S$, $H \subset N(c)$, $\forall c \in C$,



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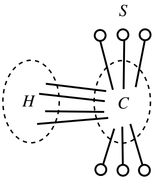

- *Thin spider*
  s_i is adjacent to $c_j \iff i = j$,
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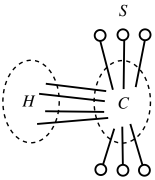

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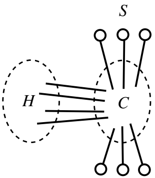

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The partition for spider and quasi-spider graphs is unique and its recognition as well as its partition can be performed in linear time (Giakoumakis, Roussel, Thuillier, 1997).

Spider graphs

Lemma

Let $G = (S, C, H)$ be a thin spider graph of weight $r \geq 2$ and T a Gtds of $G[H]$. Then:

1. $T' = (s_1, \ldots, s_r) \oplus T \oplus (c_1, \ldots, c_r)$ *is a Gtds of* G *and* $(S, C \hookleftarrow K_2, H)$. *Besides,*

$$\gamma^t_{\mathrm{gr}}(G) = \gamma^t_{\mathrm{gr}}(S, C \hookleftarrow K_2, H) = \gamma^t_{\mathrm{gr}}(G[H]) + 2r.$$

2. *If* $G[H]$ *has an isolated vertex* v*, then* $T' = (s_1, \ldots, s_{r-1}) \oplus (v) \oplus T \oplus (c_1, \ldots, c_{r-1}) \oplus (s_r) \oplus (c_r)$ *is a Gtds of* $(S \hookleftarrow K_2, C, H)$. *Otherwise,* $T' = (s_1, \ldots, s_r) \oplus T \oplus (c_1, \ldots, c_r)$ *is a Gtds of* $(S \hookleftarrow K_2, C, H)$. *Besides,*

$$\gamma^t_{\mathrm{gr}}(S \hookleftarrow K_2, C, H) = \gamma^t_{\mathrm{gr}}(G) + 2\eta(G[H]).$$

Spider graphs

Lemma

Let $G = (S, C, H)$ be a thick spider graph of weight $r \geq 3$ and Z a Gtds of $G[H]$. Then,

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   Besides,
   $$\gamma^t_{\text{gr}}(G) = \gamma^t_{\text{gr}}(S, C \hookleftarrow K_2, H) = 4 + \gamma^t_{\text{gr}}(G[H]).$$
2. $Z' = (s_1, s_2, s_r, s'_r) \oplus Z \oplus (c_1, c_2)$ *is a Gtds of* $(S \hookleftarrow K_2, C, H)$*.*
   Besides,
   $$\gamma^t_{\text{gr}}(S \hookleftarrow K_2, C, H) = \gamma^t_{\text{gr}}(G) + 2 = 6 + \gamma^t_{\text{gr}}(G[H]).$$

P_4-TIDY GRAPHS

Let U be a subset of vertices inducing a P_4 in G. A partner of U is a vertex $v \in G - U$ such that $U \cup \{v\}$ induces at least two P_4s in G. A graph G is P_4-tidy if any P_4 has at most one partner.

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THEOREM

A Gtds can be obtained in linear time for P_4-tidy graphs.

Hardness result

Grundy Total Domination Number Problem

Input: $G = (V, E)$, $k \in \mathbb{Z}^+$.

Question: $\gamma_{\mathrm{gr}}^t(G) \geq k$?

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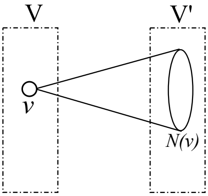

$$\gamma_{\text{gr}}^t(G') = 2\gamma_{\text{gr}}^t(G)$$

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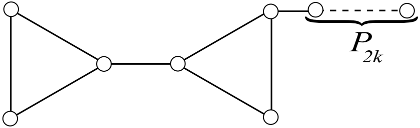
  - If $n \geq 5$ is odd,



Graph G_{5+2k}

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- Is Grundy Total Domination Number Problem an MSOL problem? (bounded clique-width)

Thanks for your attention!