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On the diclique-behavior of digraphs

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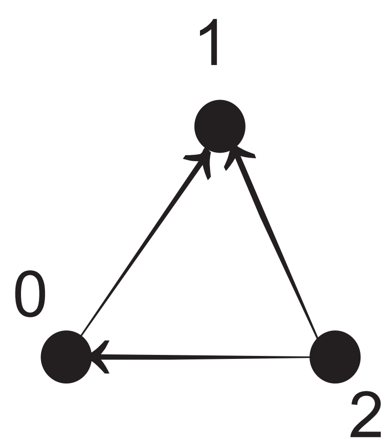
DIGRAPHS

A *digraph* $D = (V, A)$, V a non-empty finite set and $A \subseteq V \times V$.
Multiple arcs are not allowed.

Notation: $xy \in A$ or $x \longrightarrow y$

Observation: We consider irreflexive digraphs, i.e digraphs without loops.

Example: $V(D) = \{0, 1, 2\}$;
$A(D) = \{01, 20, 21\}$

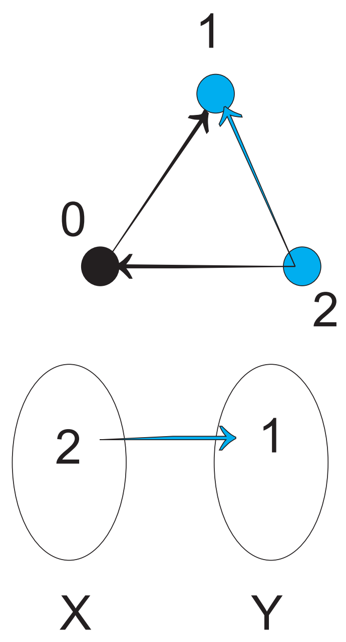


Prisner: Disimplex

$X \subseteq V$, $Y \subseteq V$, not empty sets, not necessarily disjoint.

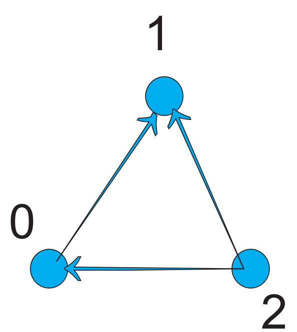
A ***disimplex*** $K(X, Y)$ of D is the subdigraph whose vertex set is $X \cup Y$ and an arc goes from every vertex of X to every vertex of Y (when $X \cap Y \neq \emptyset$, loops are not considered).

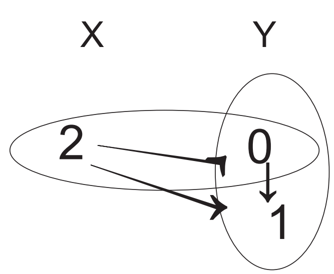
Example: $X = \{2\}$, $Y = \{1\}$. **Notation:** $\{2\} \longrightarrow \{1\}$.



Prisner: Disimplex

Observation: $\{2\} \longrightarrow \{1\} \subset \{0,2\} \longrightarrow \{0,1\}$





$\{0, 2\} \Longrightarrow \{0, 1\}$ is not subdigraph of any other disimplex.

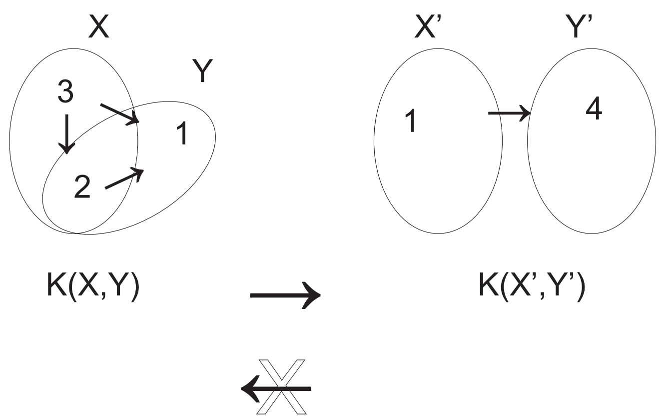
A ***diclique*** $K(X, Y)$ of D is a disimplex that is not a proper subdigraph of any other disimplex.

Prisner: Diclique operator

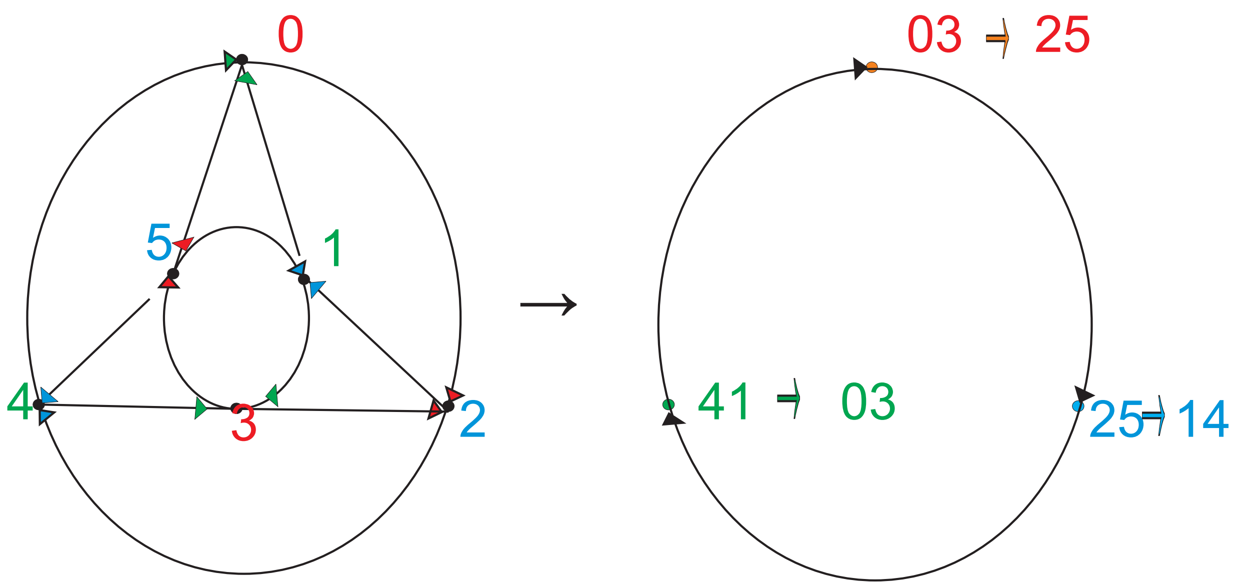
The ***diclique operator*** $\overrightarrow{k}(D)$ is defined by

$V(\overrightarrow{k}(D)) = \{K(X,Y):\ K(X,Y)\ is\ a\ diclique\ of\ D\}$ and

$A(\overrightarrow{k}(D)) = \{(K(X,Y), K(X',Y')):\ Y \cap X' \neq \emptyset\}$.



Diclique operator: Example



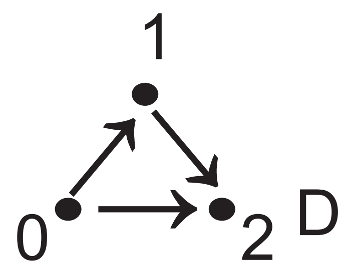
D

$\vec{k}(D)$

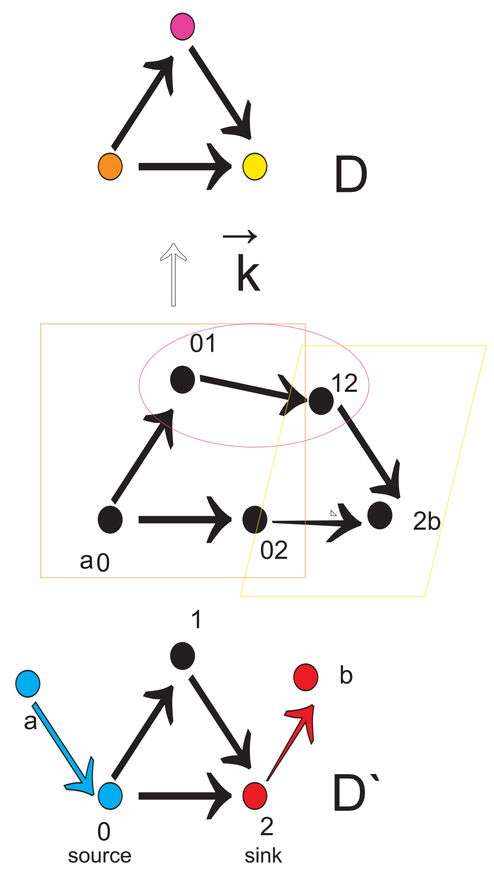
What does it know about diclique operator?

Lemma[Prisner]:

Every digraph D is the diclique digraph of some digraph.



Prisner: Diclique operator



Iterated diclique digraphs

The ***iterated*** diclique digraphs $\overrightarrow{k}^n(D)$ are defined by:

$\overrightarrow{k}^0(D) = D$ and $\overrightarrow{k}^n(D) = \overrightarrow{k}(\overrightarrow{k}^{n-1}(D))$

Iterated diclique digraphs

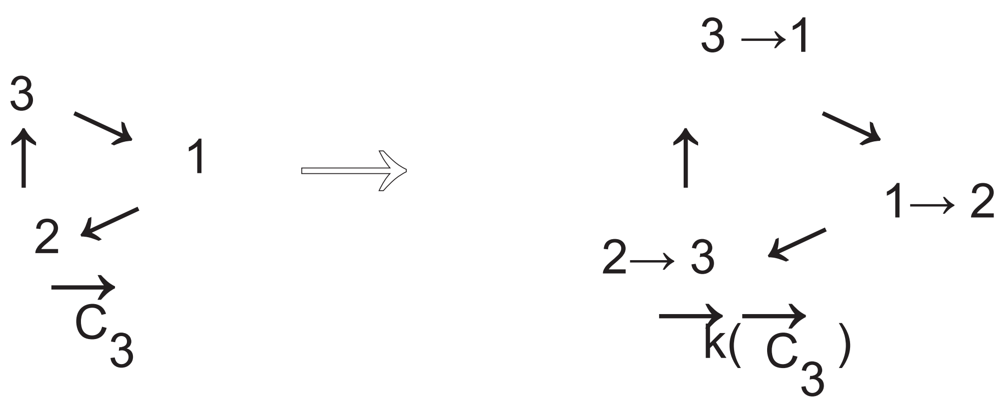
A digraph D is $\overrightarrow{k}$***-divergent*** if $|V(\overrightarrow{k}^n(D))|$ tends to infinity with n,

otherwise D is $\overrightarrow{k}$***-convergent***

Iterated diclique digraphs

A digraph D is $\overrightarrow{k}$***-periodic*** if there is $n \in N$ such that $\overrightarrow{k}^n(D) \approx D$.

If $n = 1$ then D is ***self-diclique***.



Divergence

What does it know about divergence?

What theoretical results are known?

It is known about divergence:

Lemma[Prisner]:

If the irreflexive digraph D contains some irreflexive K_n^i,
then $K_m^i \subseteq \overrightarrow{k}(D)$ with $m = \binom{n}{\lceil \frac{n}{2} \rceil}$

It is known about divergence:

Definition: An induced subdigraph H of D is called a *retract* of D if there is a retraction $r : V(D) \longrightarrow V(H)$, i.e r is a weak homomorphism such that there is $h : V(H) \longrightarrow V(D)$ such that $rh = id_{V(D)}$.

Corollary[Prisner]:

All finite digraph containing some (irreflexive) K_n^i as retract must $\overrightarrow{k}$-diverge.

Periodicity

What does it know about $\overrightarrow{k}$-periodicity?

Problem about periodicity:

Problem (suggested by Prisner): Are there, besides the directed cycles, more $\overrightarrow{k}$-periodic digraphs in the family of all finite strongly connected digraphs?

About periodicity

Answer: A self-diclique digraph different from cycle was given by Zelinka 2002: $\overrightarrow{O_3}$ denotes an Eulerian orientacion of O_3(the complement on 3 disjoint isomorphic copies of the complete K_2).

About periodicity

Answer: An infinite family of self-diclique digraphs given by Figueroa-LLano 2010: for $n \geq 5$ $\overrightarrow{C}_n((1,2))$ having $V(\overrightarrow{C}_n((1,2))) = Z_n$, $A(\overrightarrow{C}_n((1,2))) = \{(i,j) : i,j \in Z_n \, and \, j-i \in \{1,2\}\}$.

About periodicity

Answer: A characterization of the self-diclique circulant digraph and an infinite family of non-circulant self-diclique digraphs given by Frick-LLano-Zuazua 2015:

The only self-diclique circulant digraphs without symetric arcs are:
$\overrightarrow{C}_n (n \geq 3)$ and $\overrightarrow{C}_n((1,2))(n \geq 5)$.

The digraph for $m \geq 3$, D_m with $V(D_m) = Z_{2m}$ and $A(D_m) = \{(i, i+j) : i = 0, 2, 3, .., 2m-2;\ j = 1, 2, 3\} \cup \{(i, i+j) : i = 1, 3, 5, ..., 2m-1;\ j = 1, 2\}$

Open problem resect to periodicity:

Open question: Are there, besides the digraphs $\overrightarrow{C}_n (n \geq 3)$, $\overrightarrow{C}_n(1,2)(n \geq 5)$ and $D_m(m \geq 3)$ any other strong self-diclique digraphs without symmetric arcs?

Convergence

A family convergent: The Fast-Fourier-Transform digraph

It is known about convergence

Family of convergence digraphs:

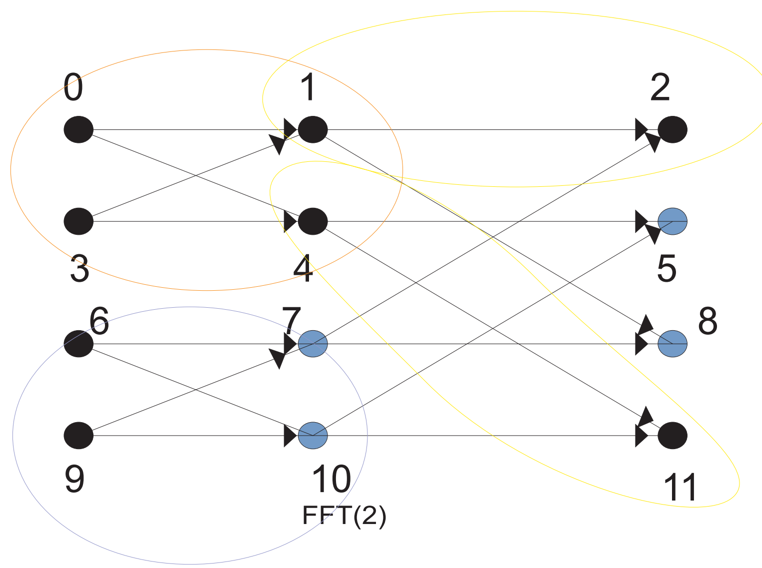
1. $FFT(n)$ with $|V(FFT(n)) = Z_{2^n(n+1)}| = 2^n(n+1)$

   [Heydemann -Sotteau]

2. All dicliques of $FFT(n)$ are $\overrightarrow{K}_{2,2}$.

All satisfying $\overrightarrow{k}(FFT(n)) = FFT(n-1)$

Example: $V(FFT(2)) = Z_{12}$,
dicliques $\{i, i+4\} \Rightarrow \{i+1, i+5\}$ $i \in \{0, 6\}$; $\{i, i+3\} \Rightarrow \{i+1, i+4\}$ with $i \in \{1, 4\}$

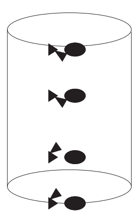


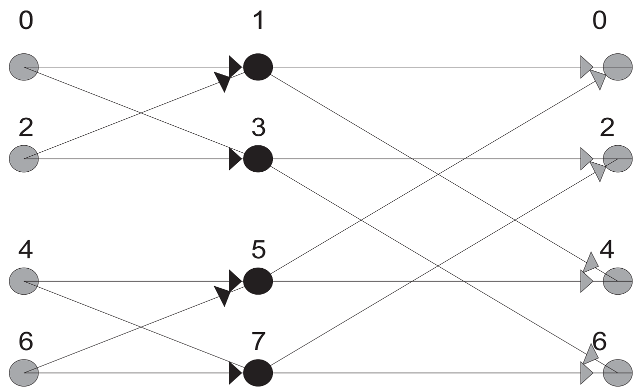
Contents

- Convergence
- Divergence

Contents: A family convergent

Convergence: Butterfly $BF(n)$ [Heydemann and D. Sotteau]



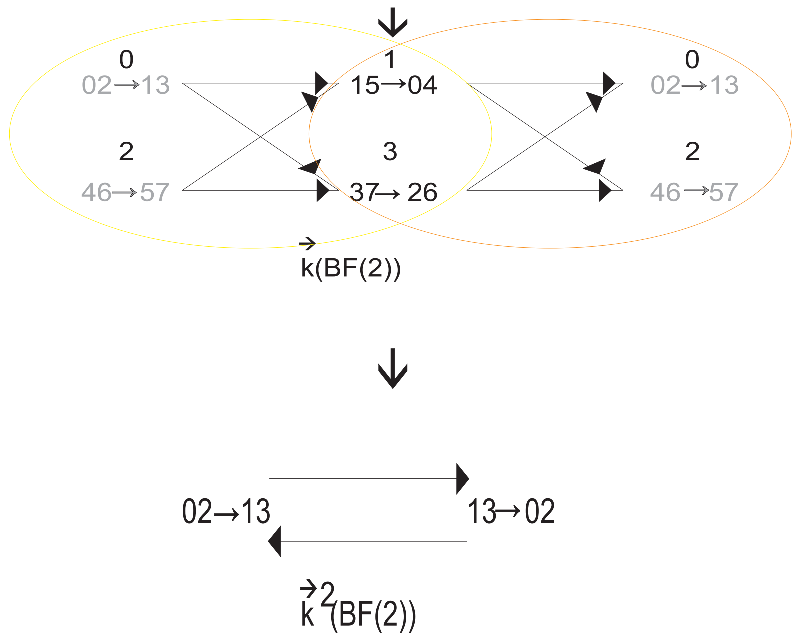


BF(2)

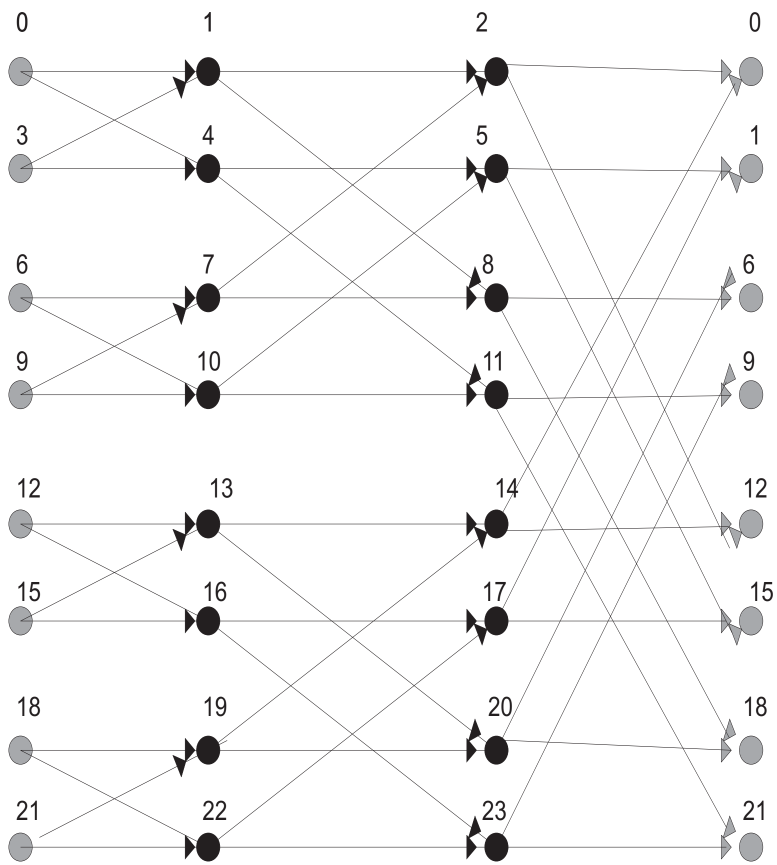
dicliques BF(2)

02→13 15→04

46→57 37→ 26



0
02→13
1
15→04
0
02→13
2
46→57
3
37→ 26
2
46→57
$\vec{k}$(BF(2))
02→13
13→02
$\vec{k}^2$(BF(2))


0
1
2
0
3
4
5
1
6
7
8
6
9
10
11
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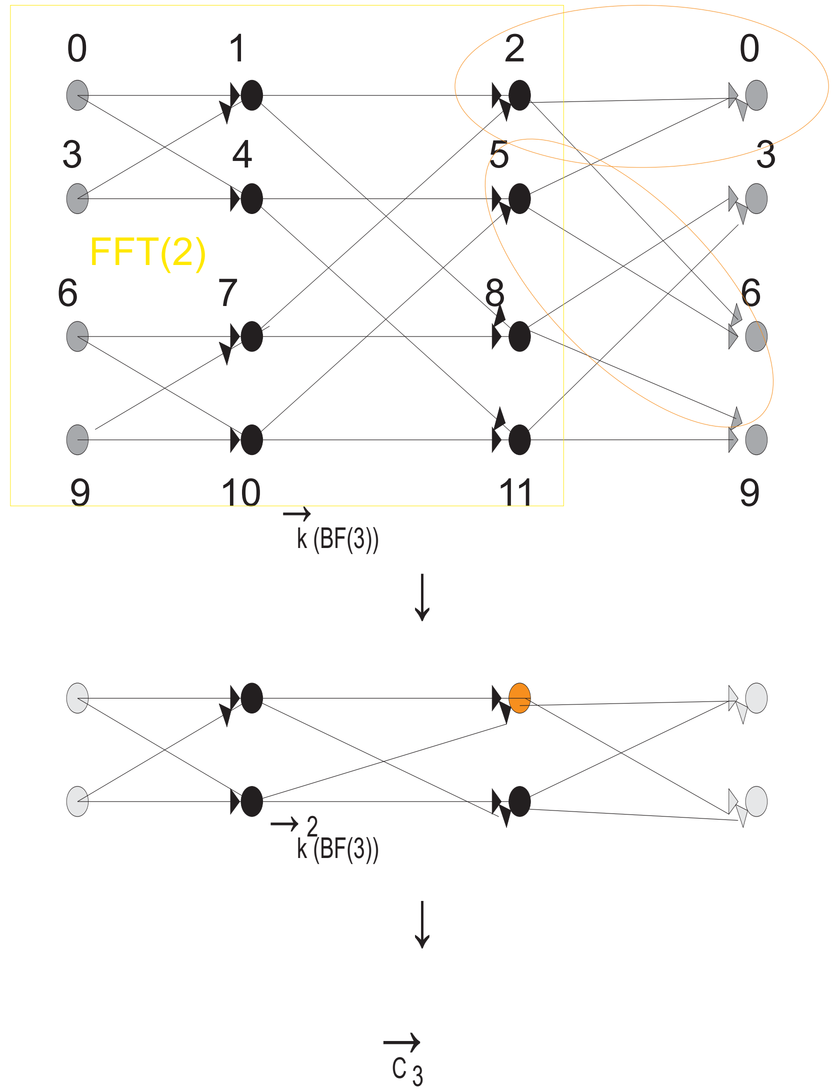
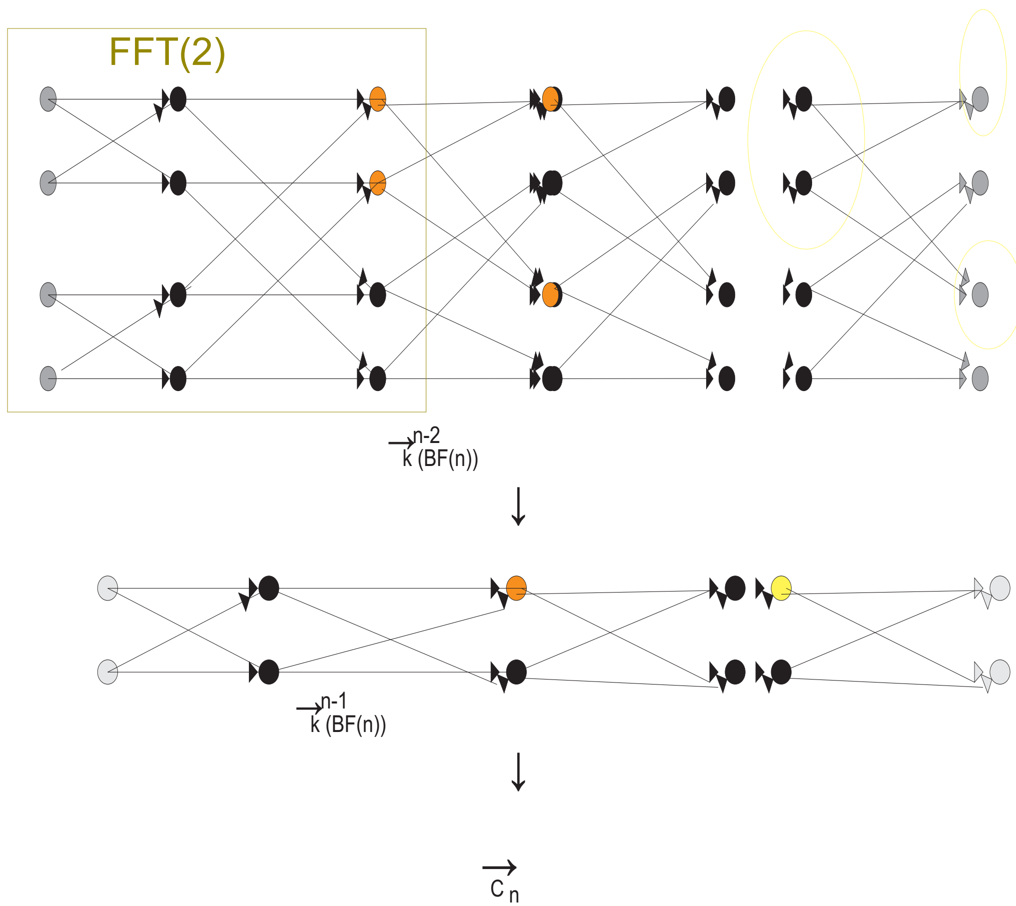
23
21

BF(3)


0
1
2
0
3
4
5
3
FFT(2)
6
7
8
6
9
10
11
9
k (BF(3))
k 2 (BF(3))
C 3

Step n-2



Convergence

Proposition

$$\overrightarrow{k}^{n}(BF(n)) \approx \overrightarrow{C}_n \text{ for every } n \geq 2.$$

Contents

Divergence: Clique vs Diclique

Clique operator: K	**Diclique operator:** $\overrightarrow{k}$
Retractions	Retractions
Octahedron	Who plays the role?
Coaffinations	?

Divergence: Irreflexive strongly connected digraphs

Binary *digraph relation* $f : D_1 \to D_2$ is a subset $V(D_1) \times V(D_2)$ such that:

1. $f(u) \neq \emptyset$ for every $u \in V(D_1)$
2. if $u =\!\!\to v$, then $u' =\!\!\to v'$ for every $u' \in f(u)$ and $v' \in f(v)$.

Divergence: Irreflexive strongly connected digraphs

Digraph relation induces a binary relation on $V(\overrightarrow{k}(D_1)) \times V(\overrightarrow{k}(D_2))$
$f_{\overrightarrow{k}}(A \Longrightarrow B) = \{A' \Longrightarrow B' \in \overrightarrow{k}(D_2) : \; f(A \Longrightarrow B) = f(A) \Longrightarrow f(B) \subseteq A' \Longrightarrow B'\}$

Proposition: $f_{\overrightarrow{k}}$ is a digraph relation. If f is digraph isomorphism then $f_{\overrightarrow{k}}$ too.

Divergence: Irreflexive strongly connected digraphs

Automorphic digraph D: (D, α)

Symmetric-preserving relation:

$f : D_1 \longrightarrow D_2$ satisfying

$f \circ \alpha = \beta \circ f$

Proposition: $f : D_1 \longrightarrow D_2$ is a symmetry-preserving relation between automorphic digraph. Then $f_{\overrightarrow{k}}$ is too.

Divergence: Irreflexive strongly connected digraphs

Automorphism coaffine: α of D

1. $\alpha(u) \neq u$ for every $u \in V(D)$

2. u, $\alpha(u)$ are not adjacent for every $u \in V(D)$

3. $\alpha(u) \notin A$ if $u \in A$ and $\alpha(u) \notin B$ if $u \in B$ for every $A \Longrightarrow B$ diclique of D

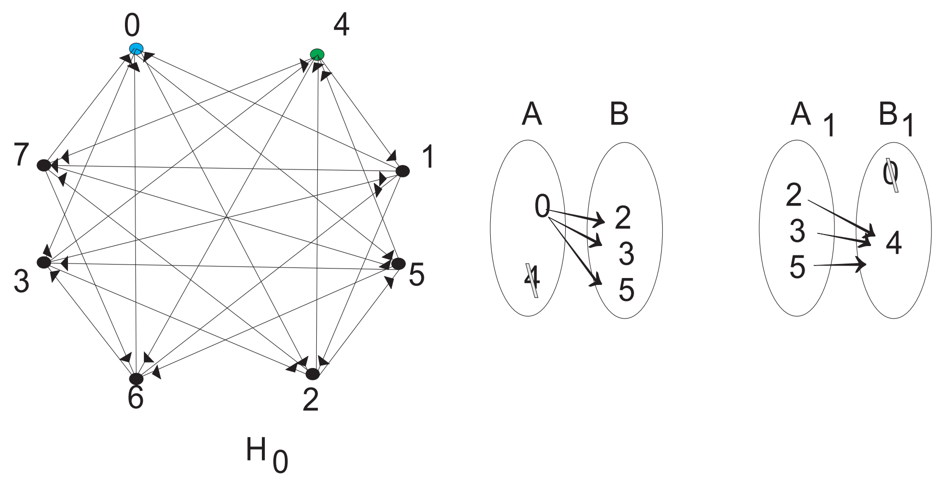
Example coaffine

[Gray-Macpherson-Praeger-Royle] H_0 with $V(H_0) = Z_8$

Coaffination of H_0:

$\alpha : Z_8 \to Z_8$ defined by $\alpha(i) = i + 4(mod\, 8)$ for $i \in \{0, 1, 2, 3, 4\}$.

α-orbits (invariants): $\{0, 4\}$, $\{1, 5\}$, $\{2, 6\}$ and $\{3, 7\}$.



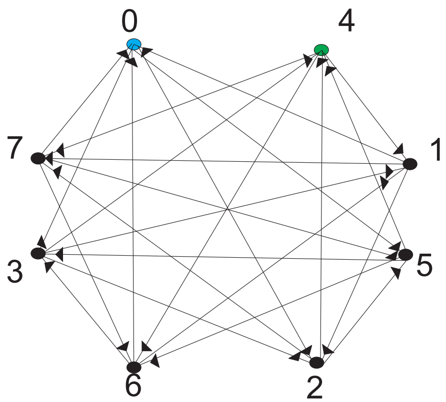
Coaffinations:

Proposition: If D $= (D, \alpha)$ is coaffine, then $\overrightarrow{k}$(D) $= (\overrightarrow{k}(D), \alpha_{\overrightarrow{k}})$ is too.

D$= (D, \alpha)$ coaffine.
rank of D (r(D)) $=$ number of α-orbits

Proposition: If $f : D_1 \to D_2$ symmetry-preserving relation between coaffine digraphs. Then $r(D_1) \leq r(D_2)$. Also if D_1 is rank divergent, D_2 is too.

Who plays the role of the octahedron?



H_0

dicliques de H_0

0	0 → 2,3,5	1,6,7 → 0	0
1	1 → 0,2,7	3,4,6 → 1	1
2	2 → 4,5,7	1,3,0 → 2	2
3	3 → 2,4,1	5,6,0 → 3	3
4	4 → 6,7,1	3,5,2 → 4	4
5	5 → 4,6,3	7,0,2 → 5	5
6	6 → 0,1,3	5,7,4 → 6	6
7	7 → 6,0,5	1,2,4 → 7	7

dicliques de $\overrightarrow{k(H_0)}$

0 ⟶ 2,3,5, 1,2,3,4,5,6,7	0 ⟶ 0, 2,3,5
1 ⟶ 0,2,7, 0,2,3,4,5,6,7,	1 ⟶ 1, 0,2,7
:	:
:	:

copies H_0

Divergence

Theorem: H_0 is $\overrightarrow{k}$-diverges

Corollary: All finite digraph containing H_0 as retract must $\overrightarrow{k}$-diverge.

Open problems:

Open question: Are there, besides the digraphs $\overrightarrow{C}_n (n \geq 3)$, $\overrightarrow{C}_n(1,2)(n \geq 5)$ and $D_m(m \geq 3)$ any other strong self-diclique digraphs without symmetric arcs?

Open question: Are there, besides retract of H_0, any other divergence digraphs?

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MUCHAS GRACIAS!