On the diclique-behavior of digraphs

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A digraph $D = (V, A)$, $V$ a non-empty finite set and $A \subseteq V \times V$. Multiple arcs are not allowed.

**Notation:** $xy \in A$ or $x \rightarrow y$

**Observation:** We consider irreflexive digraphs, i.e digraphs without loops.
Example: $V(D) = \{0, 1, 2\}$; $A(D) = \{01, 20, 21\}$
Prisner: Disimplex

\[ X \subseteq V, \ Y \subseteq V, \] not empty sets, not necessarily disjoint.

A \textit{disimplex} \( K(X, Y) \) of \( D \) is the subdigraph whose vertex set is \( X \cup Y \) and an arc goes from every vertex of \( X \) to every vertex of \( Y \) (when \( X \cap Y \neq \emptyset \), loops are not considered).
Example: $X = \{2\}$, $Y = \{1\}$. Notation: $\{2\} \rightarrow \{1\}$.
Prisner: Disimplex

Observation: $\{2\} \rightarrow \{1\} \subset \{0, 2\} \rightarrow \{0, 1\}$
A diclique $K(X, Y)$ of $D$ is a disimplex that is not a proper subdigraph of any other disimplex.
The **diclique operator** $\overrightarrow{k}(D)$ is defined by

$$V(\overrightarrow{k}(D)) = \{K(X, Y) : K(X, Y) \text{ is a diclique of } D\} \text{ and}$$

$$A(\overrightarrow{k}(D)) = \{(K(X, Y), K(X', Y')) : Y \cap X' \neq \emptyset\}.$$
Diclique operator: Example

\[ D_k(D) \]
What does it know about diclique operator?
Lemma[Prisner]:

Every digraph $D$ is the diclique digraph of some digraph.
Prisner: Diclique operator
Iterated diclique digraphs

The *iterated* diclique digraphs $\overrightarrow{k}^n(D)$ are defined by:

$$\overrightarrow{k}^0(D) = D \text{ and } \overrightarrow{k}^n(D) = \overrightarrow{k}(\overrightarrow{k}^{n-1}(D))$$
Iterated diclique digraphs

A digraph $D$ is $\overrightarrow{k}$-\textit{divergent} if $|V(\overrightarrow{k}^n(D))|$ tends to infinity with $n$, otherwise $D$ is $\overrightarrow{k}$-\textit{convergent}
Iterated diclique digraphs

A digraph $D$ is $\overrightarrow{k}$-periodic if there is $n \in \mathbb{N}$ such that $\overrightarrow{k^n}(D) \approx D$.

If $n = 1$ then $D$ is self-diclique.
Divergence

What does it know about divergence?

What theoretical results are known?
It is known about divergence:

Lemma[Prisner]:

If the irreflexive digraph $D$ contains some irreflexive $K_{n}^{i}$, then $K_{m}^{i} \subseteq \vec{k}(D)$ with $m = \left(\frac{n}{\lceil \frac{n}{2} \rceil}\right)$. 
It is known about divergence:

**Definition:** An induced subdigraph $H$ of $D$ is called a *retract* of $D$ if there is a retraction $r : V(D) \to V(H)$, i.e. $r$ is a weak homomorphism such that there is $h : V(H) \to V(D)$ such that $rh = id_{V(D)}$.

**Corollary [Prisner]:**

All finite digraph containing some (irreflexive) $K^n_1$ as retract must $\overrightarrow{k}$-diverge.
What does it know about $\vec{k}$-periodicity?
Problem about periodicity:

Problem (suggested by Prisner): Are there, besides the directed cycles, more $\vec{k}$-periodic digraphs in the family of all finite strongly connected digraphs?
About periodicity

**Answer:** A self-diclique digraph different from cycle was given by Zelinka 2002: $\overrightarrow{O_3}$ denotes an Eulerian orientation of $O_3$ (the complement on 3 disjoint isomorphic copies of the complete $K_2$).
About periodicity

**Answer:** An infinite family of self-diclique digraphs given by Figueroa-LLano 2010: for $n \geq 5$ $\overrightarrow{C_n}((1, 2))$ having $V(\overrightarrow{C_n}((1, 2))) = \mathbb{Z}_n$, $A(\overrightarrow{C_n}((1, 2))) = \{(i, j) : i, j \in \mathbb{Z}_n \text{ and } j - i \in \{1, 2\}\}$. 
About periodicity

**Answer:** A characterization of the self-diclique circulant digraph and an infinite family of non-circulant self-diclique digraphs given by Frick-LLano-Zuazua 2015:

The only self-diclique circulant digraphs without symmetric arcs are: \( \vec{C}_n(n \geq 3) \) and \( \vec{C}_n((1, 2))(n \geq 5) \).

The digraph for \( m \geq 3, D_m \) with \( V(D_m) = Z_{2m} \) and

\[
A(D_m) = \{(i, i + j) : i = 0, 2, 3, \ldots, 2m - 2; \ j = 1, 2, 3\} \cup \{(i, i + j) : i = 1, 3, 5, \ldots, 2m - 1; \ j = 1, 2\}
\]
Open problem resect to periodicity:

Open question: Are there, besides the digraphs $\vec{C}_n(n \geq 3)$, $\vec{C}_n(1, 2)(n \geq 5)$ and $D_m(m \geq 3)$ any other strong self-diclique digraphs without symmetric arcs?
Convergence

A family convergent: The Fast-Fourier-Transform digraph
It is known about convergence

Family of convergence digraphs:

1. $FFT(n)$ with $|V(FFT(n))| = Z_{2^n(n+1)} = 2^n(n+1)$

   [Heydemann -Sotteau]

2. All dicliques of $FFT(n)$ are $\overrightarrow{K}_{2,2}$.

All satisfying $\overrightarrow{k}(FFT(n)) = FFT(n - 1)$
Example: $V(FFT(2)) = Z_{12}$, dicliques $\{i, i+4\} \Rightarrow \{i+1, i+5\} i \in \{0, 6\}$; $\{i, i+3\} \Rightarrow \{i+1, i+4\}$ with $i \in \{1, 4\}$
Contents

- Convergence
- Divergence
Contents: A family convergent

Convergence: Butterfly $BF(n)$ [Heydemann and D. Sotteau]
BF(2)

dicliques BF(2)

02→13 15→04

46→57 37→26
Convergence

Proposition

\[ \overrightarrow{k}^n(BF(n)) \approx \overrightarrow{C}_n \text{ for every } n \geq 2. \]
## Divergence: Clique vs Diclique

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Divergence: Irreflexive strongly connected digraphs

Binary digraph relation \( f : D_1 \rightarrow D_2 \) is a subset \( V(D_1) \times V(D_2) \) such that:

1. \( f(u) \neq \emptyset \) for every \( u \in V(D_1) \)

2. if \( u \rightarrow v \), then \( u' \rightarrow v' \) for every \( u' \in f(u) \) and \( v' \in f(v) \).
Divergence: Irreflexive strongly connected digraphs

Digraph relation induces a binary relation on $V(\overrightarrow{k}(D_1)) \times V(\overrightarrow{k}(D_2))$

$f_{\overrightarrow{k}}(A \rightarrow\!\!\!\!\!\rightarrow B) = \{A' \rightarrow\!\!\!\!\!\rightarrow B' \in \overrightarrow{k}(D_2) : f(A \rightarrow\!\!\!\!\!\rightarrow B) = f(A) \rightarrow\!\!\!\!\!\rightarrow f(B) \subseteq A' \rightarrow\!\!\!\!\!\rightarrow B'\}$

**Proposition:** $f_{\overrightarrow{k}}$ is a digraph relation. If $f$ is digraph isomorphism then $f_{\overrightarrow{k}}$ too.
Divergence: Irreflexive strongly connected digraphs

Automorphic digraph $D$: $(D, \alpha)$

Symmetric-preserving relation:

$f : D_1 \rightarrow D_2$ satisfying

$f \circ \alpha = \beta \circ f$

Proposition: $f : D_1 \rightarrow D_2$ is a symmetry-preserving relation between automorphic digraph. Then $f \xrightarrow{k}$ is too.
Divergence: Irreflexive strongly connected digraphs

Automorphism coaffine: $\alpha$ of $D$

1. $\alpha(u) \neq u$ for every $u \in V(D)$

2. $u, \alpha(u)$ are not adjacent for every $u \in V(D)$

3. $\alpha(u) \notin A$ if $u \in A$ and $\alpha(u) \notin B$ if $u \in B$ for every $A \implies B$ diclique of $D$
Example coaffine

[Gray-Macpherson-Praeger-Royle] $H_0$ with $V(H_0) = Z_8$

Coaffination of $H_0$:

$\alpha : Z_8 \rightarrow Z_8$ defined by $\alpha(i) = i + 4 (mod 8)$ for $i \in \{0, 1, 2, 3, 4\}$.

$\alpha$-orbits (invariants): $\{0, 4\}$, $\{1, 5\}$, $\{2, 6\}$ and $\{3, 7\}$. 
Coaffinations:

**Proposition:** If $D = (D, \alpha)$ is coaffine, then $\overrightarrow{k}(D) = (\overrightarrow{k}(D), \alpha \overrightarrow{k})$ is too.

$D = (D, \alpha)$ coaffine.

*rank* of $D$ ($r(D)$) = number of $\alpha$-orbits

**Proposition:** If $f : D_1 \rightarrow D_2$ symmetry-preserving relation between coaffine digraphs. Then $r(D_1) \leq r(D_2)$. Also if $D_1$ is rank divergent, $D_2$ is too.
Who plays the role of the octahedron?
dicliques de $H_0$

0 0 $\rightarrow$ 2,3,5 1,6,7 $\rightarrow$ 0 0
1 1 $\rightarrow$ 0,2,7 3,4,6 $\rightarrow$ 1 1
2 2 $\rightarrow$ 4,5,7 1,3,0 $\rightarrow$ 2 2
3 3 $\rightarrow$ 2,4,1 5,6,0 $\rightarrow$ 3 3
4 4 $\rightarrow$ 6,7,1 3,5,2 $\rightarrow$ 4 4
5 5 $\rightarrow$ 4,6,3 7,0,2 $\rightarrow$ 5 5
6 6 $\rightarrow$ 0,1,3 5,7,4 $\rightarrow$ 6 6
7 7 $\rightarrow$ 6,0,5 1,2,4 $\rightarrow$ 7 7

dicliques de $k(H_0)$

$0 \rightarrow 2,3,5,1,2,3,4,5,6,7$ $0 \rightarrow 0,2,3,5$

$1 \rightarrow 0,2,7,0,2,3,4,5,6,7$ $1 \rightarrow 1,0,2,7$

copies $H_0$
Divergence

**Theorem:** $H_0$ is $\vec{k}$-diverges

**Corollary:** All finite digraph containing $H_0$ as retract must $\vec{k}$-diverge.
Open problems:

Open question: Are there, besides the digraphs \( \vec{C}_n(n \geq 3) \), \( \vec{C}_n(1, 2)(n \geq 5) \) and \( D_m(m \geq 3) \) any other strong self-diclique digraphs without symmetric arcs?

Open question: Are there, besides retract of \( H_0 \), any other divergence digraphs?
References:


MUCHAS GRACIAS!