VII Latin American Workshop on Cliques in Graphs

Fractional isomorphism of graphs and hipergraphs and its applications

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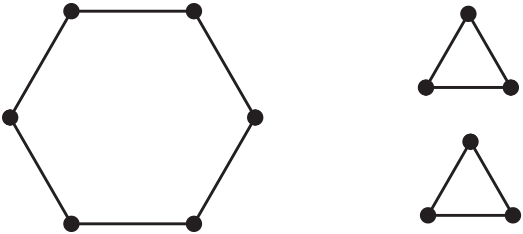
November 11, 2016

Isomorphism and fractional isomorphism of graphs

- Isomorphism of graphs: $A = PBP^{-1}$ with P a permutation matrix and A and B the adyacency matrices of G and H.
- Isomorphism of graphs: $AP = PB$ (We want to "relaxate" this equality in the fractional isomorphism).
- A matrix S is *double stochastic* if all the coefficients of S are non-negative and every row and every column of S sums 1.
- Fractional isomorphism of graphs: Graphs G and H are *fractional isomorph* ($G \approx_f H$) if there exist a matrix S double stochastic that $AS = SB$ with A and B the adyacency matrices of G and H.

An example of two fractional isomorph graphs

- $G = C_6$ and H, the disjoint union of two K_3, are graphs 2-regular of 6 vertex and are fractional isomorph.
- But C_6 (a connected graph) is not isomorph to the disjoint union of two K_3 (a disconnected graph).



An example of two fractional isomorph graphs

- $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$.
- If $S = \frac{1}{6}J_6 = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ then $AS = \frac{2}{6}J_6$ and $SB = \frac{2}{6}J_6$.

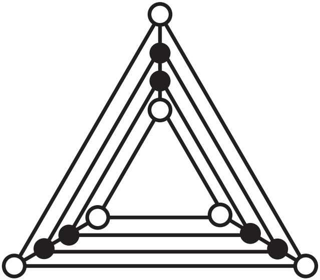
Some properties of the fractional isomorphism

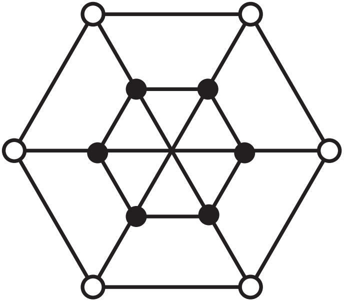
- The relation $G \approx_f H$ is an equivalence relation.
- If $G \simeq H$ then $G \approx_f H$.
- Two fractional isomorph graphs have the same number of vertex.
- Two fractional isomorph graphs have the same number of edges.
- The degrees sequencies of two fractional isomorph graphs are the same and the maximum eingenvalue also.

Equitable partitions of a graph

- Given a graph G, always is possible to divide the set of its vertex $V(G)$ in subsets $V(G) = V_1 \cup ... \cup V_s$ that every induced subgraph $G[V_i]$ is regular and every induced bipartite subgraph $G[V_i, V_j]$ is bi-regular.
- We call such a division an *equitable partition* of G.
- All graphs have an equitable partition if we consider every vertex of the graph as a part of the partition (singletons).
- If G is a regular graph then a possible equitable partition is consider only one part $V(G)$.
- Given any graph G, it has an unique coarsest equitable partition.

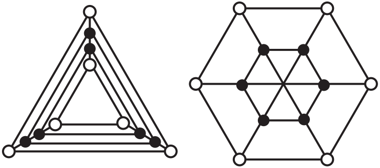
Equitable partitions of a graph





Parameter matrix for the equitable partitions

- Given an equitable partition of a graph G, we calculate the parameters (n, D) of G: if we have the partition $V(G) = V_1 \cup \ldots \cup V_s$ then n is a vector of s entries and D is a $s \times s$ matrix.
- The graphs of the figure have the same parameters: $n = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.
- G y H are not isomorph: G is planar and H not. But, ¿are they fractional isomorph?



The main theorem for the fractional isomorphism

The main theorem for the fractional isomorphism caracterize the fractional isomorph graphs.

Theorem

Given two graphs G and H, the follow are equivalent:

1. $G \approx_f H$
2. G and H have in common the coarsest equitable partition
3. G and H have in common some equitable partition
4. $D(G) = D(H)$

Isomorphism of hipergraphs and a "relaxation"

- If two hipergraphs G and H are isomorph then $PM_GQ = M_H$ with M_G and M_H the vertex-edge incidence matrix of G and H and P and Q two appropiate permutations matrices.
- The last equation could be written as $PM_G = M_HQ^t$ or as $M_GQ = P^tM_H$.
- ¿What happens if we "relaxate" P and Q to be double stochastic?
- For two hipergraph, we write $G \equiv H$ iff there exist S_1 and S_2 double stochastic matrices such that $S_1M_G = M_HS_2^t$ and $M_GS_2 = S_1^tM_H$.

Fractional isomorphism of hipergraphs

Theorem

- $\equiv$ is an equivalence relation that preserve the usual isomorphism of hipergraph.
- For G and H graphs (hipergraphs 2-uniform), if $G \equiv H \Rightarrow G \approx_f H$.
- For G and H graphs (hipergraphs 2-uniform), if $G \approx_f H \Rightarrow G \equiv H$.

A last comment

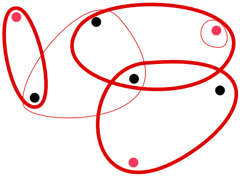
The last proof caracterize the fractional isomorphism of hipergraphs.

Theorem

Two hipergraphs G and H are fractional isomorph if it is possible to part its vertices in subsets (induced hipergraphs) such as every subset (an hipergraph) be regular and uniform with the same parameters in G and H and the hiperedges that join subsets have the same parameters in G and in H.

Covering and packing of an hipergraph

- A *covering* of an hipergraph G is a family of hiperedges X_1, ..., X_j of G such that $V(G) \subseteq X_1 \cup \ldots \cup X_J$.
- The minimum number j that it is possible to find a covering of G is called the *covering number* of G and we write $k(G)$.
- A *packing* of an hipergraph G is a subset of $Y \subseteq V(G)$ with the property that no two element of Y are in the same hiperedge of G. (In a packing, we choose some number of vertices representing hiperedges).
- The *packing number* $p(G)$ is the maximum number of elements of a packing of G.



Integer and fractional problems

- The problem of finding the covering number of G can be formulate as a minimization integer problem.
- The problem of finding the packing number of G can be formulate as a maximization integer problem.
- These problems can be "relaxated" to find $k_f(G)$ (*fractional covering number*) and $p_f(G)$ (*fractional packing number*).
- It is possible to show that $p(G) \leq p_f(G) = k_f(G) \leq k(G)$.

Fractional covering and packing for fractional isomorph hipergraphs

Theorem

- $k_f(H) = k_f(G)$ if G and H are two fractional isomorph hipergraphs.
- Thus, $p_f(H) = p_f(G)$ if G and H are two fractional isomorph hipergraphs.

Applications to the fractional matching

- A matching in a graph is a set of disjoint edges.
- The *matching number* $\mu(G)$ is the number of edge of the biggest matching.
- A *fractional matching* is a function f that asign to every edge of the graph a number in $[0,1]$ such that for every vertex v we have that $\sum f(e) \leq 1$ where the sum is over every edge incident on v.
- The *fractional matching number* $\mu_f(G)$ is the supremum of $\sum f(e)$ over all the possibles fractional matchings f.
- It is possible to show that $\mu_f(G) = p_f(H'_G)$.

Applications to the fractional matching

- What can we say about the fractional matching number of G and H if $G \approx_f H$?
- $\mu_f(G) = \mu_f(H)$.
- Perfect fractional matching: A graph G has a *perfect fractional matching* if $\mu_f(G) = \frac{1}{2}v(G)$.
- If G has a perfect fractional matching and H is fractional isomorph to G then H also has a perfect fractional matching.

Conclusions

- The fractional isomorphism of graphs appears as a "relaxation" of the isomorphism of graphs: we ask $AS = SB$ with S a double stochastic matrix.
- The fractional isomorphism of graphs can be completely caracterize with the notion of equitable partitions.
- The fractional isomorphism of graphs can be extended to hipergraphs.
- The fractional covering number (and thus, the fractional packing number) is invariant under the fractional isomorphism of hipergraphs.

Thank you!