Cliques and Graph Classes

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Standard definitions:

clique in a graph: a set of pairwise adjacent vertices

$\omega(G)$ = the clique number of G
= the maximum size of a clique in G

an **independent set** (or: a **stable set**):
a set of pairwise non-adjacent vertices
= a clique in the complement of the graph, $\overline{G}$

$\alpha(G)$ = the independence number of G
= the maximum size of an independent set in G

maximal (clique, independent set) = inclusion-wise maximal

graph class = a property of graphs closed under isomorphism

Many **graph classes** can be defined by imposing conditions
that must be satisfied
by some or all **cliques** of the graph
and/or by some or all **independent sets** of the graph.

Examples:

- **partition conditions:**

  existence of a partition of the graph's vertex set into a specified number of cliques and/or independent sets

  - k-colorable graphs
  - split graphs
  - (a, b)-graphs

- existence of a clique or independent set satisfying certain properties

  - graphs with a dominating clique
  - graphs with a clique cutset
  - graphs with a stable cutset
  - graphs with a strong clique

- **threshold conditions:**

  existence of a linear weight function on the vertices separating the stable sets (or the maximal stable sets) from all other vertex subsets

  - threshold graphs
  - equistable graphs

- **size conditions:**

  restrictions on the relations between the sizes of maximal cliques or maximal independent sets

  - triangle-free graphs ($\omega(G) \leq 2$)
  - well-covered graphs
  - graphs satisfying $\alpha(G) \cdot \omega(G) \geq |V(G)|$

- **intersection conditions:**

  restrictions related to the intersections between cliques and independent sets

  - CIS graphs
  - almost CIS graphs

- **covering conditions:**

  existence of a collection of cliques and/or independent sets covering all vertices or edges and satisfying certain conditions

  - simplicial graphs
  - edge simplicial graphs
  - weakly CIS graphs
  - normal graphs

- and many more ...

Some of the conditions are **hereditary**, that is, if the condition holds for a graph, then it holds for all its induced subgraphs.

For any condition, the hereditary variant of the condition can be obtained by requiring the condition
for all induced subgraphs of the graph.

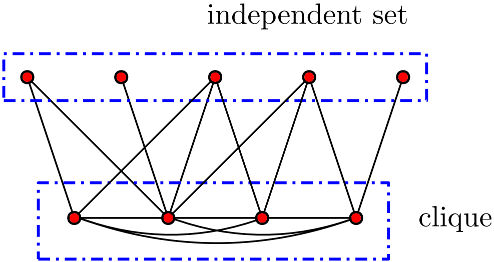
Some of the conditions are **self-complementary**, that is, if the condition holds for G, then it also holds for $\overline{G}$.

For any condition, a self-complementary variant of the condition can be obtained by requiring the condition
for both G **and** $\overline{G}$
(or: **for at least one of** G **and** $\overline{G}$).

Three well known examples

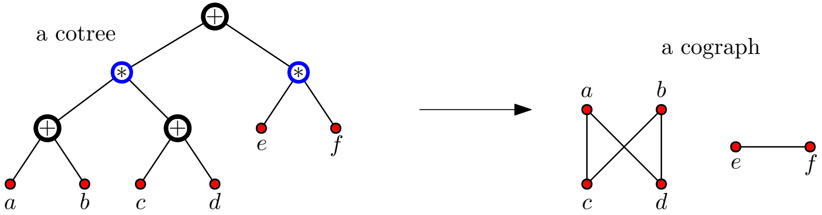
1. Split graphs [Földes, Hammer, 1977]

A graph is **split** if has a partition of its vertex set into a clique and an independent set.



2. Cographs [Corneil, Lerchs, Stewart-Burlingham, 1981]

The class of **cographs** is the smallest class of graphs containing K_1 that is closed under disjoint union and join.



A graph is a cograph if and only if in each induced subgraph, each maximal clique intersects each maximal stable set.

3. Perfect graphs [Berge, early 1960s]

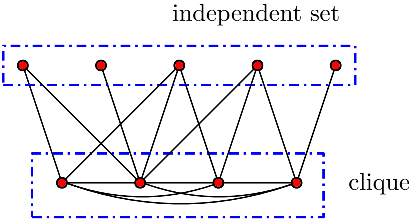
A graph G is **perfect** if each induced subgraph H of G has $\chi(H) = \omega(H)$.

Theorem (Lovász, 1972)

A graph is perfect if and only if each induced subgraph H of G satisfies $\alpha(H)\omega(H) \geq |V(H)|$.

Back to split graphs

Split graphs



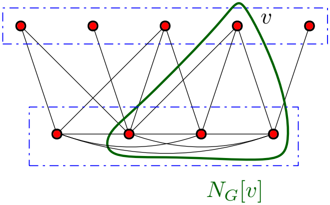
Many characterizations of split graphs are known.
Split graphs can be recognized in linear time.

Many generalizations of split graphs were studied in the literature (including previous editions of LAWCG).

Simplicial cliques

A vertex v in a graph G is **simplicial** if its closed neighborhood $N_G[v]$ is a clique. Any such clique is **simplicial**.

In a split graph, each vertex in the independent set is simplicial:



Consequently:

Every vertex of a split graph is in a simplicial clique.

What if we require that **every edge** of the graph is in a simplicial clique?

Cheston, Hare, Hedetniemi, and Laskar, 1988:
edge simplicial graphs

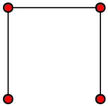
Edge simplicial graphs coincide with **upper bound graphs**, introduced by **McMorris and Zaslavsky** in 1982.

- A graph is **upper bound** if there exists a **partially ordered set** $(P,\leq)$ such that $V(G)=P$ and two distinct vertices x and y are adjacent if and only if the set $\{x,y\}$ has an upper bound in P.

Edge simplicial graphs can be recognized in polynomial time.

Edge simplicial and split graphs

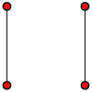
Not all split graphs are edge simplicial: P_4 is not.



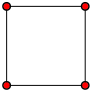
P_4

While the class of split graphs is **self-complementary**, the class of edge simplicial graphs is not:

- $2K_2$ is edge simplicial, but its complement, C_4, is not.



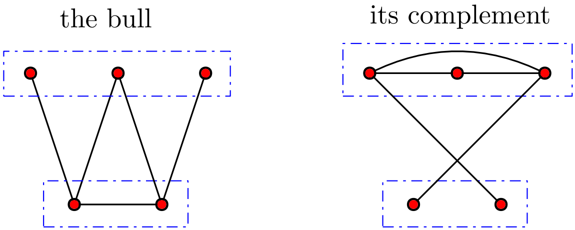
$2K_2$



C_4

What are the graphs G such that **both G and $\overline{G}$ are edge simplicial**?

Example: the bull (a self-complementary split graph)



Theorem (Boros, Gurvich, M., 2015+)

If both G end $\overline{G}$ are edge simplicial, then G is split.

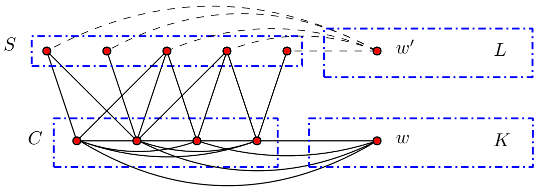
Proof idea:

$\{C_1, \ldots, C_k\}$: the set of simplicial cliques of G,
u_i a simplicial vertex in C_i
$S = \{u_1, \ldots, u_k\}$

$\{S_1, \ldots, S_\ell\}$: the set of simplicial cliques of $\overline{G}$,
v_j a simplicial vertex in S_j (in $\overline{G}$),
$C = \{v_1, \ldots, v_\ell\}$

Then S is a stable set and C is a clique in G.

Every vertex $w \in V(G) \setminus (C \cup S)$ satisfies either $C \subseteq N(w)$ or $N(w) \cap S = \emptyset$.



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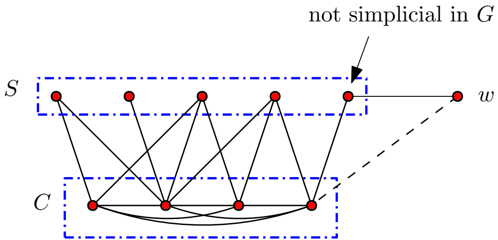

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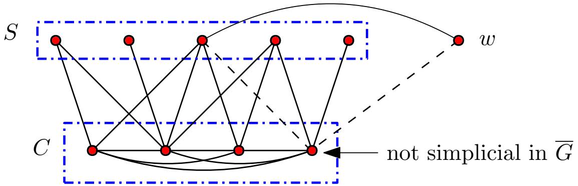

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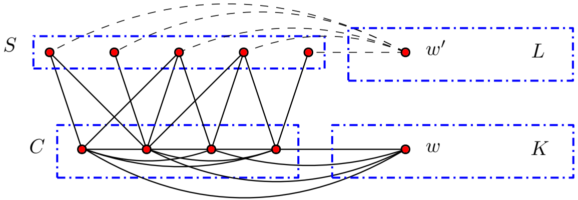


Let

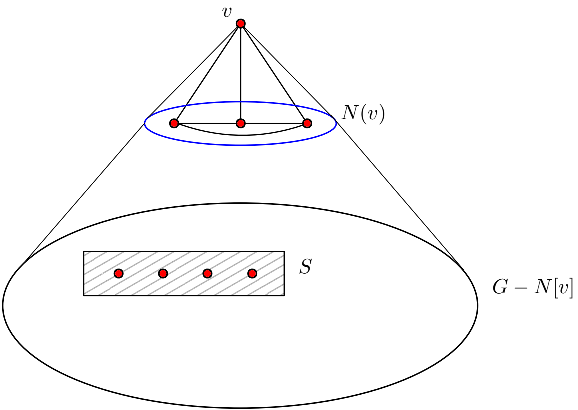
$$
\begin{aligned}
K &= \{w \in V(G) \setminus (C \cup S) : C \subseteq N(w)\} \\
L &= \{w \in V(G) \setminus (C \cup S) : N(w) \cap S = \emptyset\}
\end{aligned}
$$

Using the fact that $\overline{G}$ is edge simplicial, it can be shown that K is a clique in G. Similarly, L is a stable set in G.

$\Longrightarrow$ G is split. $\square$

A more general concept: strong cliques

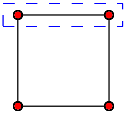
Every simplicial clique **intersects all maximal stable sets** of the graph.



A clique C in a graph is **strong** if it **intersects all maximal stable sets** of the graph.

Not every strong clique in a graph is simplicial.

- Every edge of the 4-cycle is a strong clique, but not a simplicial one.



C_4

- A clique C is **not simplicial** if and only if every vertex of C has a neighbor outside C $\Longleftrightarrow$ $N(C)$ dominates C.

- A clique C is **not strong** $\Longleftrightarrow$ $\exists$ stable set $S \subseteq N(C)$ that dominates C.

Proposition (Hujdurović, M., Ries, 2016+)

Every strong clique in a C_4-free graph is simplicial.

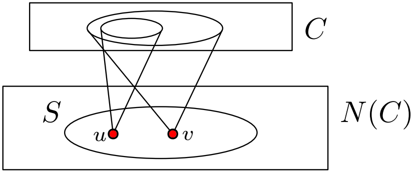
Proof:

Let C be a strong clique that is not simplicial.

Since C is not simplicial, $N(C)$ dominates C.

Let S be any inclusion-minimal subset of $N(C)$ that dominates C.

By minimality, every two vertices in S have incomparable neighborhoods in C.



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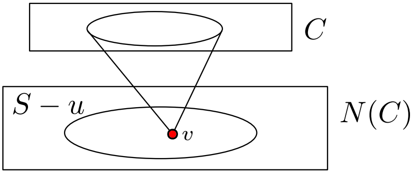

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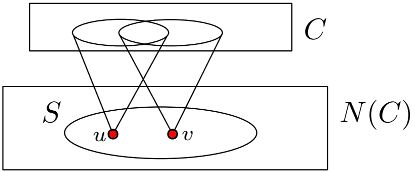

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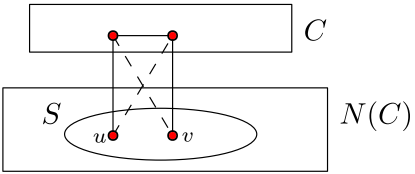


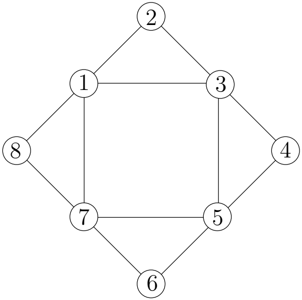
G is C_4-free $\Longrightarrow$ set S is stable

$\Longrightarrow$ C is not strong, a contradiction $\square$

What are the graphs in which **every maximal clique is strong**?

A graph G is **CIS (Clique-Intersects-Stable)**
if every maximal clique C and every maximal stable set S intersect, that is, $C \cap S \neq \emptyset$.

Example: any graph in which every maximal clique is simplicial

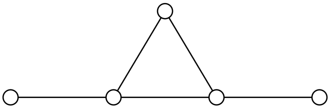


	2468	15	37	146	368	582	724
123	2	1	3	1	3	2	2
345	4	5	3	4	3	5	4
567	6	5	7	6	6	5	7
781	8	1	7	1	8	8	7

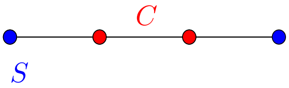
Source: D. V. Andrade, E. Boros, and V. Gurvich, On Graphs Whose Maximal Cliques and Stable Sets Intersect, RRR 17-2006, RUTCOR Research Reports, Rutgers Univ.

CIS graphs are not hereditary

The *bull* is CIS:



However, P_4 is not:



Every P_4-free graph is CIS.

CIS graphs were studied by Grillet (1969), Berge (1985), Zang (1995), and Andrade, Boros, and Gurvich (2006).

In 2015, Dobson, Hujdurović, M., and Verret characterized CIS graphs among the **vertex-transitive** graphs as those such that:

- G is well-covered (all maximal stable sets have the same size),
- $\overline{G}$ is well-covered, and
- $\alpha(G) \cdot \omega(G) = |V(G)|$.

Every vertex-transitive graph satisfies $\alpha(G) \cdot \omega(G) \leq |V(G)|$.

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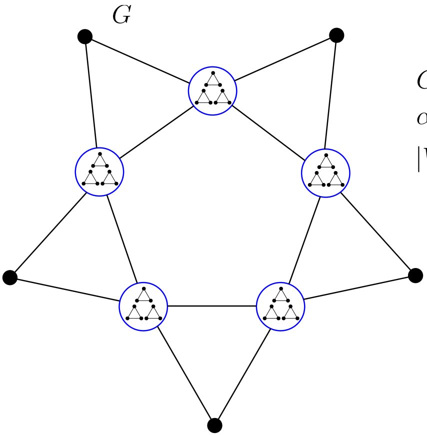
In the same paper, Dobson et al. asked if every CIS graph satisfies $\alpha(G) \cdot \omega(G) \geq |V(G)|$.

This question was resolved in the negative very recently, by Liliana Alcón, Marisa Gutierrez, and M., in October 2016 here in **La Plata**.



G is CIS

$\alpha(G) = \omega(G) = 7$

$|V(G)| = 50$

No good characterization of CIS graphs is known.

The complexity of recognizing CIS graphs is open.

The problem is conjectured to be

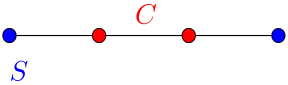
- **co-NP-complete** by Zverovich, Zverovich (2006);
- **in P** by Andrade, Boros, Gurvich (2006).

Generalizations of CIS graphs

Definition (Boros, Gurvich, Zverovich, 2009)

A graph G is **almost CIS** if every maximal clique C and every maximal stable set S intersect, except for a unique pair.

Example: P_4



Boros, Gurvich, Zverovich conjectured that **every almost CIS graph is split**.

Theorem (Wu, Zang, Zhang, 2009)

The almost CIS graphs coincide with the split graphs with a unique split partition.

In particular, almost CIS graphs can be recognized in polynomial time.

Recall:

Theorem (Boros, Gurvich, M., 2015+)

If both G end $\overline{G}$ are edge simplicial, then G is split.

The above two results imply the following characterization of split graphs:

Theorem (Boros, Gurvich, M., 2015+)

A graph G is split if and only if one of the following conditions holds:

- *G is almost CIS, or*
- *both G and $\overline{G}$ are edge simplicial.*

Semi-weakly CIS graphs

A graph is **semi-weakly CIS** if every edge of G is contained in a strong clique.

- This terminology is due to Boros, Gurvich, M. (2015).

  (The paper is to appear in a special issue of DAM dedicated to the 65th birthday of Andreas Brandstädt.)

The complexity of recognizing semi-weakly CIS graphs is open.

A pair of set families $\mathcal{A}$, $\mathcal{B}$ is said to be **cross-intersecting** if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$.

$\mathcal{S}(G)$: the family of all maximal stable sets of G

A graph G is **semi-weakly CIS** $\Longleftrightarrow$
$\exists$ a collection $\mathcal{C}$ of maximal cliques covering all edges
such that the pair $\mathcal{C}$, $\mathcal{S}(G)$ is cross-intersecting.

More generally:

A graph G is **weakly CIS** if and only if
it has a cross-intersecting pair $\mathcal{C}$, $\mathcal{S}$ where
$\mathcal{C}$ is a collection of maximal cliques covering all edges
and
$\mathcal{S}$ is a collection of maximal stable sets covering all non-edges.

Also introduced by Boros, Gurvich, M. in 2015.

The complexity of recognizing weakly CIS graphs is open.

Even more generally **(edges, non-edges $\rightsquigarrow$ vertices)**:

A graph G is **normal** if and only if
it has a cross-intersecting pair $\mathcal{C}$, $\mathcal{S}$ where
$\mathcal{C}$ is a collection of maximal cliques covering all vertices
and
$\mathcal{S}$ is a collection of maximal stable sets covering all vertices.

Introduced by Körner in 1973 and studied in a series of papers.

- Every perfect graph is normal.
- **Normal Graph Conjecture** (De Simone, Körner, 1999):

  Every $\{C_5, C_7, \overline{C_7}\}$-free graph is normal.
- A disproof of the conjecture was announced in March 2016 by Harutyunyan, Pastor, and Thomassé.

Back to semi-weakly CIS graphs

In 1993, McAvaney, Robertson, and DeTemple showed that the semi-weakly CIS graphs coincide with the so-called

general partition graphs, introduced essentially already by DeTemple, Robertson, and Harary in 1984:

A graph $G = (V, E)$ is a **general partition graph** if G is the intersection graph of a set system over a finite ground set U such that every maximal stable set of G corresponds to a partition of U.

$\{2\}$

$\{1\}$ $\{1,2\}$ $\{2,3\}$ $\{3\}$

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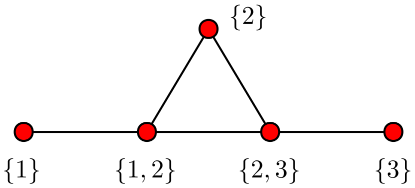

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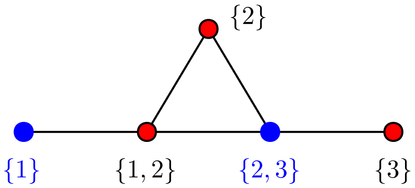

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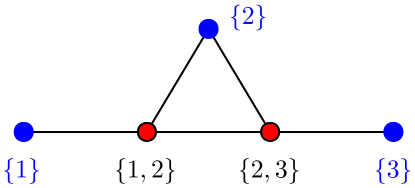

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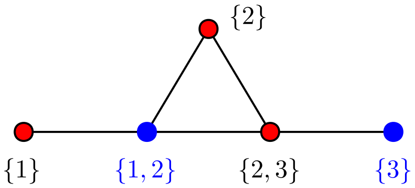


Semi-weakly CIS graphs are further generalized by the **equistable graphs**.

Definition (Payan, 1980)

A graph $G = (V, E)$ is **equistable** if there exists a function $w : V \to \mathbb{R}_+$ such that

$\forall S \subseteq V$:

$$S \text{ is a maximal stable set in } G \iff w(S) = \sum_{v \in S} w(v) = 1\,.$$

The complexity of recognizing equistable graphs is open.

Rise and fall of three conjectures on equistable graphs

In 2009, Jim Orlin proved that every general partition graph is equistable and conjectured that the converse holds as well.

In 1994, Mahadev, Peled, and Sun introduced the class of so-called **strongly equistable** graphs.

They proved that every strongly equistable graph is equistable and conjectured that the converse holds.

Orlin's result together with some results by Mahadev, Peled, Sun implies that every general partition graph is strongly equistable:

$$\begin{array}{c} \text{equistable} \\ \cup\!| \\ \text{strongly equistable} \\ \cup\!| \\ \text{general partition} \\ \| \\ \text{semi-weakly CIS} \end{array}$$

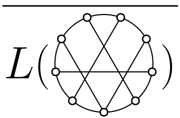
Thus, Orlin's conjecture would imply the conjecture of Mahadev, Peled, and Sun.

An intermediate **conjecture** (Miklavič and M., 2011):
Every equistable graph has a strong clique.

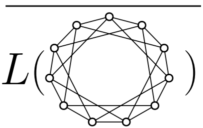
The three conjectures were proved to hold in several graph classes. However, recently all three conjectures were disproved (M., Trotignon, 2016).

The counterexamples were found within the class of **complements of line graphs of triangle-free graphs**.

equistable

$\supsetneq$ ← 

strongly equistable

$\supsetneq$ ← 

general partition

$=$

semi-weakly CIS

The main idea of our approach

Suppose that G is a triangle-free graph.
We "translate" the essential concepts from $\overline{L(G)}$ to G.

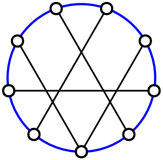
$\overline{L(G)}$	$L(G)$	G
maximal stable set	maximal clique	maximal star
edge	non-edge	a pair of disjoint edges
strong clique	strong stable set	perfect matching*
equistable	"equicliqueable"	equistarable
semi-weakly CIS		2-extendable*

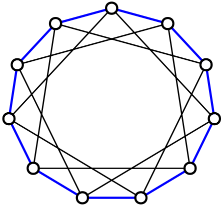
For *, we also need that $\delta(G) \geq 2$.

In 1980, Plummer defined a graph to be k**-extendable** if it contains a matching of size k and every matching of size k is contained in a perfect matching.

We then develop a sufficient condition for $\overline{L(G)}$ to be equistable.

The condition is expressed in terms of existence of an odd Hamiltonian cycle in G having chords with particular properties.





Such graphs G are not 2-extendable since they do not have any perfect matching $\iff$ $\overline{L(G)}$ is not semi-weakly CIS.

Three open questions

Orlin's (disproved) conjecture is equivalent to asking whether every equistable graph is semi-weakly CIS.

Question (Boros, Gurvich, M., 2015+)

Is the following weaker form of Orlin's conjecture true?
Every equistable graph is weakly CIS.

Question (M., Trotignon, 2016)

Does Orlin's conjecture hold within the class of perfect graphs?

Conjecture (Mahadev, Peled, Sun, 1994)

The class of strongly equistable graphs is closed under substitution.

Equistable graphs generalize the well studied class of **threshold graphs** introduced by Chvátal and Hammer in 1974.

Several variants of threshold graphs were studied in the literature.

(Benzaken and Hammer 1978, Mahadev and Peled, 1995, Chiarelli and M., 2014-16, Boros, Gurvich, M., 2015)

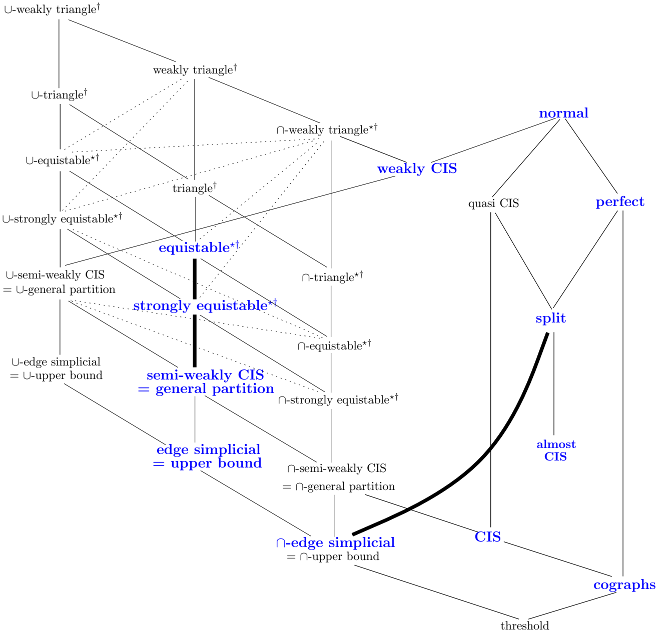
All are closely related to **threshold Boolean functions**,
or equivalently, to **threshold hypergraphs**.

(Chow, 1965, Elgot, 1965, Crama and Hammer, 2011, Golumbic, 1980, Reiterman et al., 1985)

- fruitful connections between graphs, hypergraphs, and Boolean functions (in both directions)

Threshold hypergraphs can be recognized in poly time using LP.

The existence of a "purely combinatorial" poly time algorithm for recognizing threshold hypergraphs is an **open problem** (Crama and Hammer, 2011).



Source: E. Boros, V. Gurvich, M. Milanič, On equistable, split, CIS, and related classes of graphs, DAM, to appear.

Techniques for disproving inclusions:

- probabilistic constructions
- explicit constructions
  (based on line graphs, on projective planes, etc.)

Further related variants

1) Graphs in which **every vertex is in a strong clique**

(NP-hard to recognize, Hujdurović, M., Ries, 2016+).

2) Graphs such that **in every induced subgraph of** G**, each vertex is in a strong clique**.

- Their complements were studied under the name **very strongly perfect graphs** and coincide with the so-called **Meyniel graphs**. (Burlet and Fonlupt, 1984, Hoàng, 1987).
- Polynomially recognizable.

3) **Localizable graphs** = graphs having a partition of $V(G)$ into strong cliques

- introduced by Yamashita, Kameda in 1999
- studied by Hujdurović, M., Ries, 2016+
- NP-hard to recognize
- localizable $\Longrightarrow$ well-covered

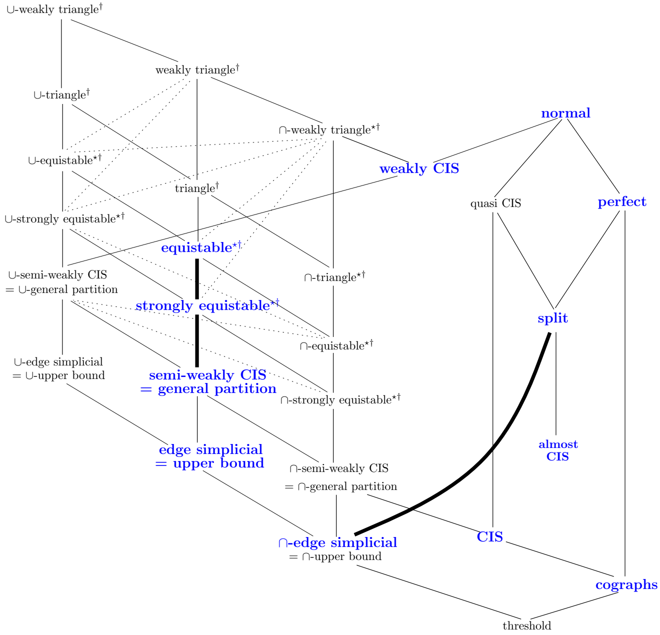
  (all maximal stable sets have the same size)
- for perfect graphs, localizable $\Longleftrightarrow$ well-covered.

4) **Graphs with a strong clique.**

- NP-hard to recognize (Hoàng, 1994).

5) Graphs **each induced subgraph of which has a strong clique**.

- Their complements are known as **strongly perfect graphs** (Berge and Duchet, 1984).
- The recognition complexity is open.



Source: E. Boros, V. Gurvich, M. Milanič, On equistable, split, CIS, and related classes of graphs, DAM, to appear.

Questions?

¡MUCHAS GRACIAS!

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LA PLATA, ARGENTINA 2016