On the P_3-Hull Number of the Cartesian Product of Graphs

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Joint work with
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Outline

Introduction

Motivation

The spread of disease on a square grid [Bollobás (2006)].

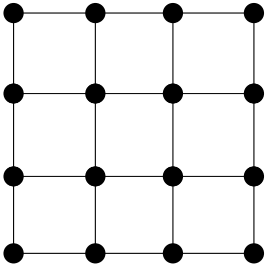

Figure 1.1: 4×4 Grid.

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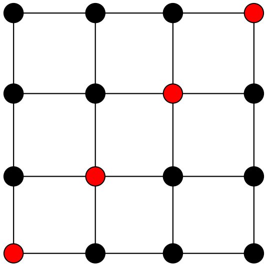

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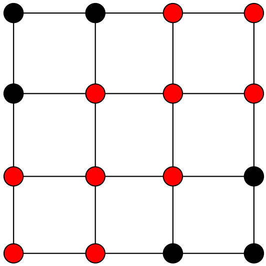

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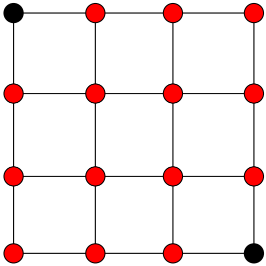

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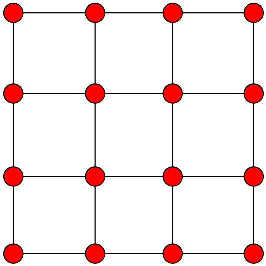

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The P_3-convexity on a graph G

We consider only finite, simple and undirected graphs. Let G be such a graph with vertex set $V(G)$. Given a set $S \subseteq V(G)$:

- Define the P_3-*interval* $I[S]$ as the set S with the set of vertices in $V(G) \setminus S$ with at least two neighbors in S.
- If $I[S] = S$, then the set S is P_3-*convex*.
- The P_3-*convex hull* $H(S)$ of S is the smallest P_3-convex set containing S.
- If $H(S) = V(G)$ we say that S is a P_3-*hull set* of G.
- The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3-*hull number* of G.

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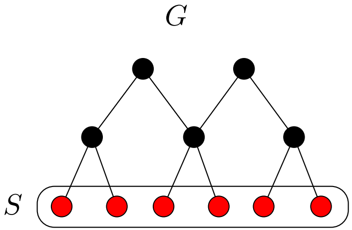

Figure 1.2: Another set $S \subseteq V(G)$.

Introduction

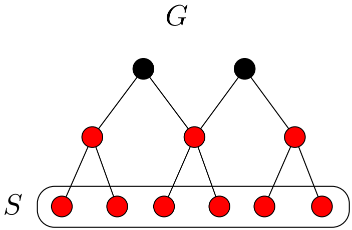

Figure 1.2: The P_3-interval $I[S]$ of S.

Introduction

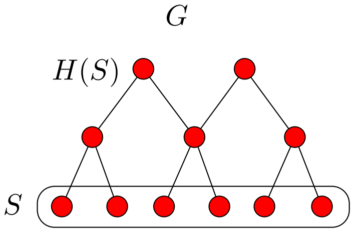

Figure 1.2: The P_3-convex hull $H(S) = V(G)$. $h(G) = 6$.

Introduction

Related Work

- [Bollobás (2006)] determined the P_3-hull number in grids $m \times n$: $h(P_m \square P_n) = \lceil \frac{m+n}{2} \rceil$.
- [Centeno et al. (2011)] proved that, given a graph G and an integer k, to decide whether $h(G) \leq k$ is NP-complete.
- [Duarte et al. (2015)]: the P_3-hull number can be determined in polynomial time for complementary prisms.

Our Aim

- We present lower and upper bounds for the P_3-hull number of the Cartesian product, $G \square H$, of general graphs G and H;
- We determine the P_3-hull number of the Cartesian product $G \square K_n$.

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More Definitions

Cartesian product $G \square H$

The graph with vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$.

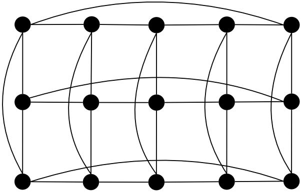


Figure 2.1: The Cartesian product $C_3 \square C_5$.

More Definitions

Line $\mathcal{L}_i$

Let G and H two graphs with vertex sets $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(H) = \{v_1, v_2, ..., v_n\}$, respectively. We refer to *line* $\mathcal{L}_i$ as the subset of vertices $\{(u_i, v_1), (u_i, v_2), \ldots, (u_i, v_n)\}$ of $V(G \square H)$.

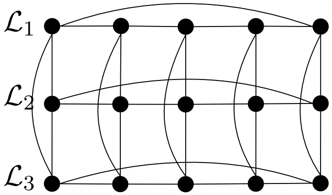


Figure 2.2: The lines $\mathcal{L}_1$ to $\mathcal{L}_3$ in the graph $C_3 \square C_5$.

More Definitions

Column $\mathcal{C}_j$

Let G and H two graphs with vertex sets $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(H) = \{v_1, v_2, ..., v_n\}$, respectively. We refer to *column* $\mathcal{C}_j$ the subset of vertices $\{(u_1, v_j), (u_2, v_j), \ldots, (u_m, v_j)\}$ of $V(G \Box H)$.

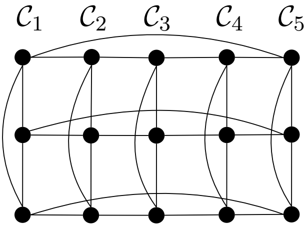


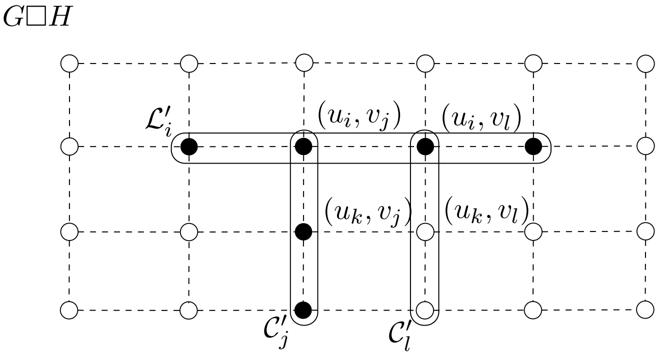
Figure 2.3: The columns $\mathcal{C}_1$ to $\mathcal{C}_5$ in the graph $C_3 \Box C_5$.

Results

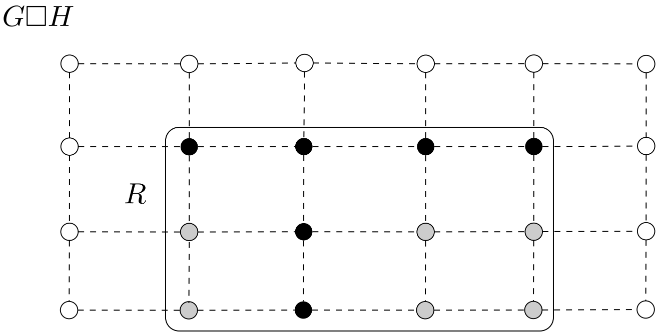
Lemma 1

Let G and H be nontrivial connected graphs, $S \subseteq V(G \square H)$ and an integer $p \geq 0$. Let $\mathcal{L}'_i \subseteq \mathcal{L}_i$, for some $i \in \{1, \ldots, m\}$ and $\mathcal{C}'_j \subseteq \mathcal{C}_j$, for some $j \in \{1, \ldots, n\}$, such that $\mathcal{L}'_i$ and $\mathcal{C}'_j$ induce connected graphs and $\mathcal{L}'_i \cap \mathcal{C}'_j \neq \emptyset$. Let $R = \{(u_k, v_l) \in V(G \square H) : (u_k, v_j) \in \mathcal{C}'_j$ and $(u_i, v_l) \in \mathcal{L}'_i\}$. If $(\mathcal{L}'_i \cup \mathcal{C}'_j) \subseteq I^p[S]$, then $R \subseteq H(S)$.

Results


$G \square H$
$\mathcal{L}'_i$
(u_i, v_j)
(u_i, v_l)
(u_k, v_j)
(u_k, v_l)
$\mathcal{C}'_j$
$\mathcal{C}'_l$

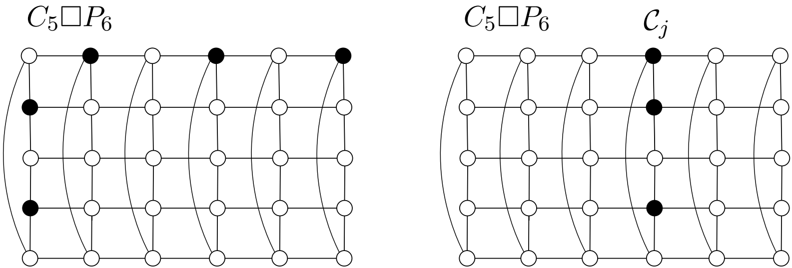
Results



Results

Projection

We call *projection* of the set $S \subseteq V(G \Box H)$ over the column $\mathcal{C}_j$, $j \in \{1, \ldots, n\}$, the set formed by the vertices $S^{\mathcal{C}_j} = \{(u_k, v_j) \in V(G \Box H) : (u_k, v) \in S, \text{ for any } v\}$.



(a) A set $S \subseteq V(C_5 \Box P_6)$.

(b) Projection of S over the column $\mathcal{C}_j$.

Results

Lemma (Projection)

Let G and H be nontrivial connected graphs and $S \subseteq V(G \Box H)$. If $H(S) = V(G \Box H)$, then $H(S^{\mathcal{C}_j}) = \mathcal{C}_j$, $j \in \{1, \ldots, n\}$.

Results

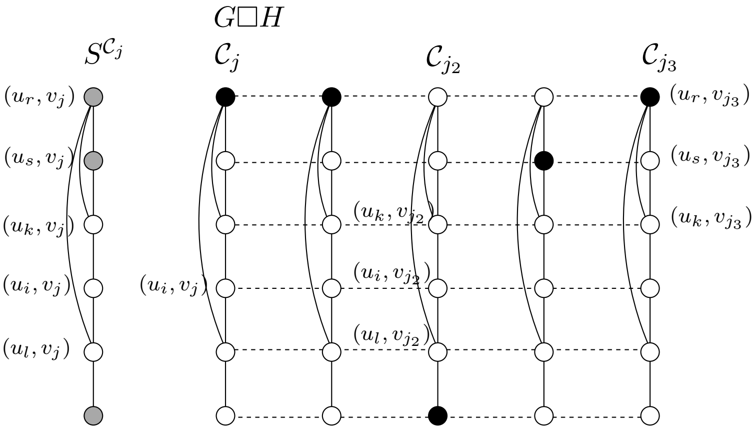


Figure 2.4: Projection of S over the column $\mathcal{C}_j$.

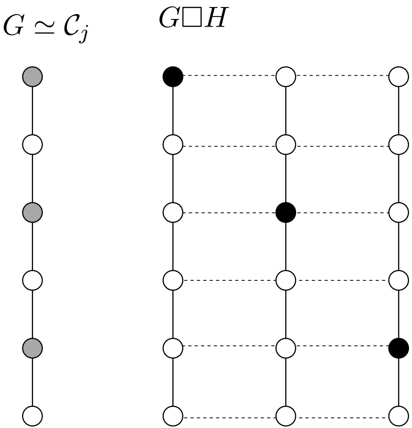
Results

Theorem (Lower Bound)

Let G and H be nontrivial connected graphs. Then $h(G\Box H) \geq max\{h(G), h(H)\}$.

Results

By contradiction, suppose that $h(G \square H) < max\{h(G), h(H)\}$.
Suppose that $h(G) \geq h(H)$. This way, $h(G \square H) < h(G)$.



Results

Type 1

Let G be a connected graph. The graph G is of the Type 1, if there exists a minimum P_3-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets A and B, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$.

Type 1a

Let G be a connected graph. The graph G is of the Type 1a, if there exists a minimum P_3-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets A and B, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$ and $|A| = 1$.

Results

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Results

Theorem (Upper Bounds)

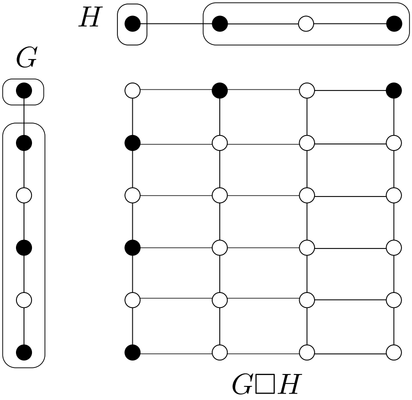
Let G and H be nontrivial connected graphs. Then:

$$h(G\Box H) \leq \begin{cases} h(G) + h(H) - 2, & \text{if } G \text{ and } H \text{ are of the Type 1a;} \\ h(G) + h(H) - 1, & \text{otherwise.} \end{cases}$$

Results

If G and H are of the Type 1a:

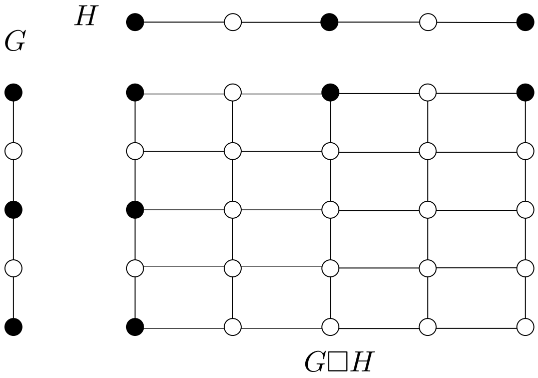
$$h(G\square H) \leq h(G) + h(H) - 2.$$



Results

Othewise (If G or H are not of the Type 1a):

$$h(G\Box H) \leq h(G) + h(H) - 1.$$



Results

Theorem

Let G be a nontrivial connected graph. Then,

$$h(G\square K_n) = \begin{cases} h(G), & \text{if } G \text{ is of the Type 1;} \\ h(G)+1, & \text{otherwise.} \end{cases}$$

Results

If G is of the Type 1:
By Theorem Lower Bound,

$$h(G\square K_n) \geq h(G).$$

Results

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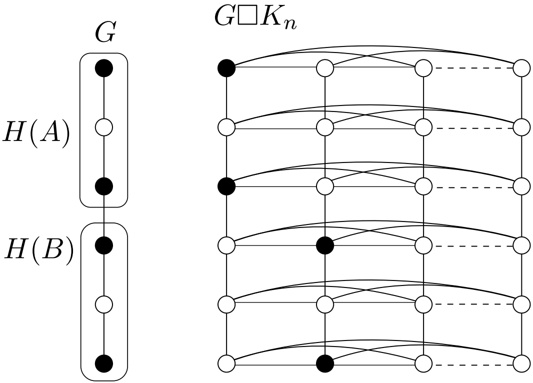

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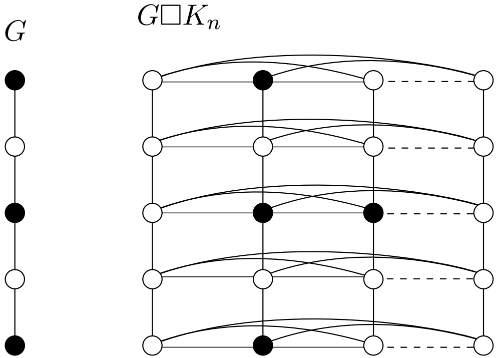
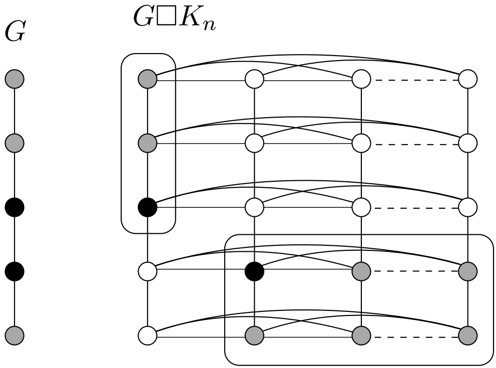
Otherwise (If G is not of the Type 1):



$$h(G\Box K_n) \leq h(G) + 1.$$

Results

Otherwise (If G is not of the Type 1):
By contradiction, suppose that $h(G \Box K_n) < h(G) + 1$.



$$h(G \Box K_n) \geq h(G) + 1.$$

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Any Questions?