On the $P_3$-Hull Number of the Cartesian Product of Graphs

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Outline

1. Introduction
   • Motivation

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   • Lower and Upper Bounds
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Motivation

The spread of disease on a square grid [Bollobás (2006)].

Figure 1.1: $4 \times 4$ Grid.
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Figure 1.1: 4 × 4 Grid.
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The $P_3$-convexity on a graph $G$

We consider only finite, simple and undirected graphs. Let $G$ be such a graph with vertex set $V(G)$. Given a set $S \subseteq V(G)$:

- Define the $P_3$-interval $I[S]$ as the set $S$ with the set of vertices in $V(G) \setminus S$ with at least two neighbors in $S$.
- If $I[S] = S$, then the set $S$ is $P_3$-convex.
- The $P_3$-convex hull $H(S)$ of $S$ is the smallest $P_3$-convex set containing $S$.
- If $H(S) = V(G)$ we say that $S$ is a $P_3$-hull set of $G$.
- The cardinality $h(G)$ of a minimum $P_3$-hull set in $G$ is called the $P_3$-hull number of $G$. 

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Figure 1.2: Another set $S \subseteq V(G)$.
Figure 1.2: The $P_3$-interval $I[S]$ of $S$. 
Introduction

Figure 1.2: The $P_3$-convex hull $H(S) = V(G)$. $h(G) = 6$. 
Introduction

Related Work

- [Bollobás (2006)] determined the $P_3$-hull number in grids $m \times n$: $h(P_m \Box P_n) = \lceil \frac{m+n}{2} \rceil$.
- [Centeno et al. (2011)] proved that, given a graph $G$ and an integer $k$, to decide whether $h(G) \leq k$ is NP-complete.
- [Duarte et al. (2015)]: the $P_3$-hull number can be determined in polynomial time for complementary prisms.

Our Aim

- We present lower and upper bounds for the $P_3$-hull number of the Cartesian product, $G \Box H$, of general graphs $G$ and $H$;
- We determine the $P_3$-hull number of the Cartesian product $G \Box K_n$. 
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More Definitions

Cartesian product $G \square H$

The graph with vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$.

Figure 2.1: The Cartesian product $C_3 \square C_5$. 
Let $G$ and $H$ two graphs with vertex sets $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$, respectively. We refer to line $\mathcal{L}_i$ as the subset of vertices $\{(u_i, v_1), (u_i, v_2), \ldots, (u_i, v_n)\}$ of $V(G \Box H)$.

Figure 2.2: The lines $\mathcal{L}_1$ to $\mathcal{L}_3$ in the graph $C_3 \Box C_5$. 
More Definitions

Column $C_j$

Let $G$ and $H$ two graphs with vertex sets $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(H) = \{v_1, v_2, ..., v_n\}$, respectively. We refer to column $C_j$ the subset of vertices $\{(u_1, v_j), (u_2, v_j), \ldots, (u_m, v_j)\}$ of $V(G \Box H)$.

![Diagram](image)

**Figure 2.3:** The columns $C_1$ to $C_5$ in the graph $C_3 \Box C_5$. 
Lemma 1

Let $G$ and $H$ be nontrivial connected graphs, $S \subseteq V(G \square H)$ and an integer $p \geq 0$. Let $L'_i \subseteq L_i$, for some $i \in \{1, \ldots, m\}$ and $C'_j \subseteq C_j$, for some $j \in \{1, \ldots, n\}$, such that $L'_i$ and $C'_j$ induce connected graphs and $L'_i \cap C'_j \neq \emptyset$. Let

$$R = \{(u_k, v_l) \in V(G \square H) : (u_k, v_j) \in C'_j \text{ and } (u_i, v_l) \in L'_i\}.$$ 

If $(L'_i \cup C'_j) \subseteq I^p[S]$, then $R \subseteq H(S)$. 
Results

\[ G \square H \]

\[ \mathcal{L}'_i \quad (u_i, v_j) \quad (u_i, v_l) \]

\[ C'_j \quad (u_k, v_j) \quad (u_k, v_l) \]

\[ C'_l \]
$G \square H$

$R$
Results

Projection

We call projection of the set $S \subseteq V(G \Box H)$ over the column $C_j$, $j \in \{1, \ldots, n\}$, the set formed by the vertices

$$S^{C_j} = \{(u_k, v_j) \in V(G \Box H) : (u_k, v) \in S, \text{ for any } v\}.$$
Lemma (Projection)

Let $G$ and $H$ be nontrivial connected graphs and $S \subseteq V(G \Box H)$. If $H(S) = V(G \Box H)$, then $H(S^{C_j}) = C_j$, $j \in \{1, \ldots, n\}$. 
Figure 2.4: Projection of $S$ over the column $C_j$. 
Theorem (Lower Bound)

Let $G$ and $H$ be nontrivial connected graphs. Then
\[ h(G \square H) \geq \max\{h(G), h(H)\}. \]
By contradiction, suppose that $h(G \square H) < \max\{h(G), h(H)\}$. Suppose that $h(G) \geq h(H)$. This way, $h(G \square H) < h(G)$.

\[ G \cong C_j \]

\[ G \square H \]
Type 1

Let $G$ be a connected graph. The graph $G$ is of the Type 1, if there exists a minimum $P_3$-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets $A$ and $B$, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$.

Type 1a

Let $G$ be a connected graph. The graph $G$ is of the Type 1a, if there exists a minimum $P_3$-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets $A$ and $B$, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$ and $|A| = 1$. 
Type 1

Let $G$ be a connected graph. The graph $G$ is of the Type 1, if there exists a minimum $P_3$-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets $A$ and $B$, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$.

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Let $G$ be a connected graph. The graph $G$ is of the Type 1a, if there exists a minimum $P_3$-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets $A$ and $B$, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$ and $|A| = 1$. 
Theorem (Upper Bounds)

Let $G$ and $H$ be nontrivial connected graphs. Then:

$$h(G \Box H) \leq \begin{cases} h(G) + h(H) - 2, & \text{if } G \text{ and } H \text{ are of the Type 1a;} \\ h(G) + h(H) - 1, & \text{otherwise.} \end{cases}$$
If $G$ and $H$ are of the Type 1a:

$$h(G \Box H) \leq h(G) + h(H) - 2.$$
Otherwise (If $G$ or $H$ are not of the Type 1a):

$$h(G \Box H) \leq h(G) + h(H) - 1.$$
Let $G$ be a nontrivial connected graph. Then,

$$h(G \Box K_n) = \begin{cases} h(G), & \text{if } G \text{ is of the Type 1;} \\ h(G) + 1, & \text{otherwise.} \end{cases}$$
If $G$ is of the Type 1:
By Theorem Lower Bound,

$$h(G \square K_n) \geq h(G).$$
If $G$ is of the Type 1:

\[ h(G \Box K_n) \leq h(G). \]
Otherwise (If $G$ is not of the Type 1):

$$h(G \square K_n) \leq h(G) + 1.$$
Otherwise (If $G$ is not of the Type 1):
By contradiction, suppose that $h(G \square K_n) < h(G) + 1$.

$h(G \square K_n) \geq h(G) + 1$. 
References

Bollobás, B. (2006)
The art of mathematics: Coffee time in Memphis
Cambridge University Press.

Irreversible conversion of graphs

Complexity properties of complementary prisms
Any Questions?