Characterizing Probe Unit Interval Graphs within the Class of Interval Graphs

Luciano Norberto Grippo

Instituto de Ciencias, Universidad Nacional de General Sarmiento

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Overview

Some definitions

- Let $\mathcal{F}$ be a finite family of non-empty sets. The intersection graph of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if their corresponding sets intersect.

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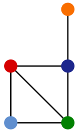
- An interval graph is the intersection graph of a finite family of closed intervals on the real line (such a family of intervals is called an interval model for the graph).

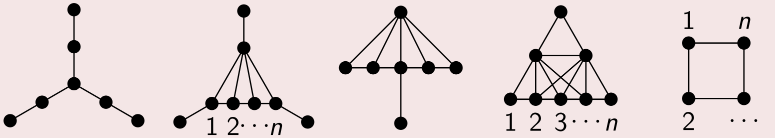


Characterization of Interval Graphs

Theorem (Boland and Lekkerkerker, 1962)

Given a graphs G, the following conditions are equivalent:

1. The graph G is an interval graph.
2. The graph G is chordal and contains no asteroidal triple. (*)
3. The graph G does not contain any of the following graphs as induced subgraphs.



bipartite claw, n-net, $n \geq 2$, umbrella, n-tent, $n \geq 3$, C_n, $n \geq 4$

- (*) An asteroidal triple is a set of three vertices such that two of them are connected by a path avoiding the neighborhood of the third one.

Unit interval graphs and Proper interval graphs

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Theorem (Roberts, 1969)

Given an interval graph G the following conditions are equivalent:

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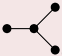
2. G is a unit interval graph
3. G contains no induced claw .

Probe Classes

- Let $G = (V, E)$ be a graph. A probe-$\mathcal{G}$ graph is a graph whose vertex set can be partitioned into two sets: a set P of probe vertices and an independent set N (a set of pairwise nonadjacent vertices) of nonprobe vertices in such a way that a graph $G^* = (N \cup P, E \cup F)$ in $\mathcal{G}$ can be obtained by adding a set F (possibly empty) of edges with both endpoints in N. Such a graph G^* is called a probe-$\mathcal{G}$ completion of G.

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- If a partitioned graph $G = (P \cup N, E)$ has a probe probe-$\mathcal{G}$ completion under this partition, $G = (P \cup N, E)$ is called a partitioned probe-$\mathcal{G}$ graph.
- Probe unit interval graphs and probe proper interval graphs are the same class. For probe interval (unit) graphs we call the vertices in P (resp. N) probe intervals (resp. nonprobe intervals) indistinctly.

Previous works

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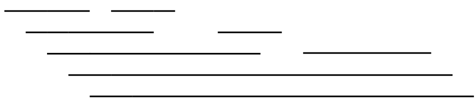
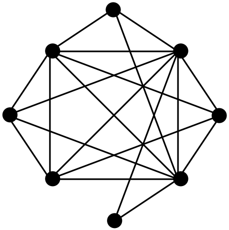
Threshold graphs

- A threshold graph is a graph with no P_4, C_4 and $2K_2$ as forbidden induced subgraph.
- Threshold graphs are interval graphs but they are not necessary unit interval graph.

Theorem (M.C. Golumbic and A. Trenk, 2004)

Threshold graphs are probe unit interval graphs.





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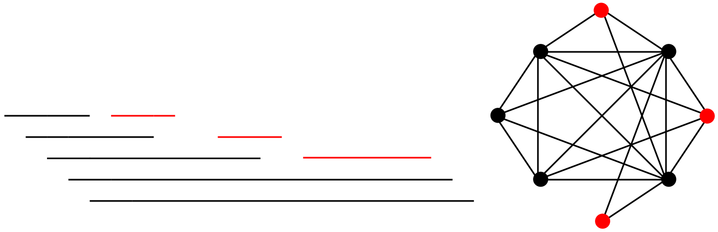

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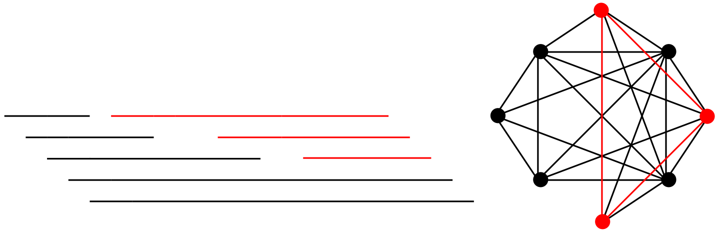

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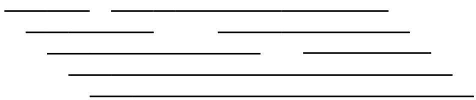
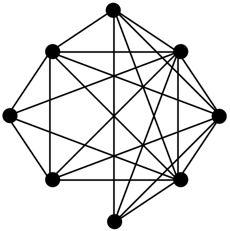
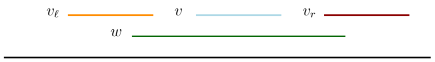
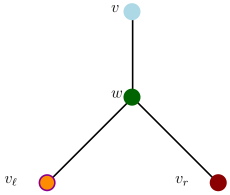




Special intervals and edges

- A vertex v of an interval graph with an interval model $\mathcal{I}$ is called a centered vertex under $\mathcal{I}$ or simply a centered vertex if there is an interval $I_w \in \mathcal{I}$ containing properly its corresponding interval $I_v \in \mathcal{I}$ and there are two vertices v_ℓ and v_r, adjacent to v, so that I_{v_ℓ} is completely to the left of I_v and I_{v_r} is completely to the right of I_v.





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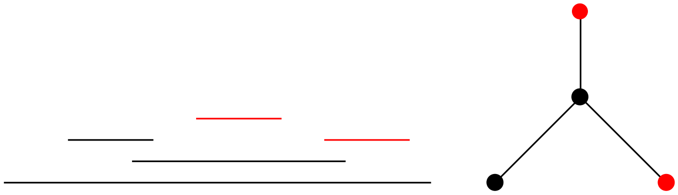

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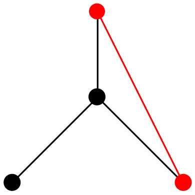

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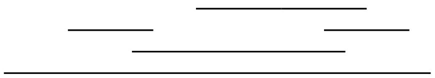
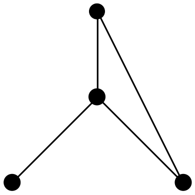
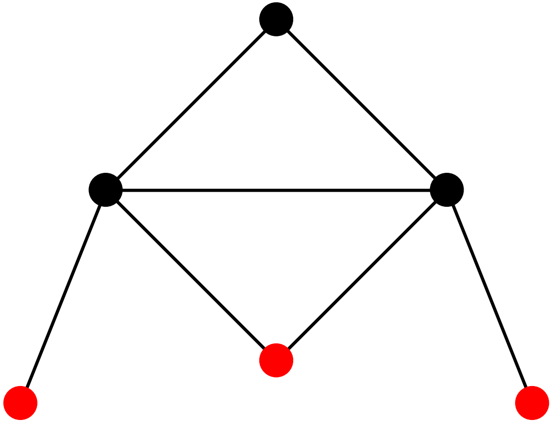




Minimal forbidden partitioned probe unit interval graph



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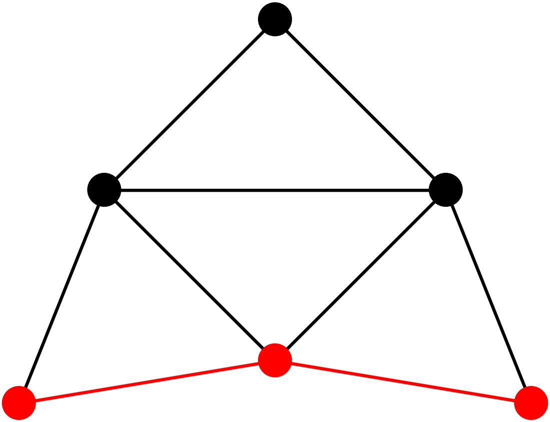

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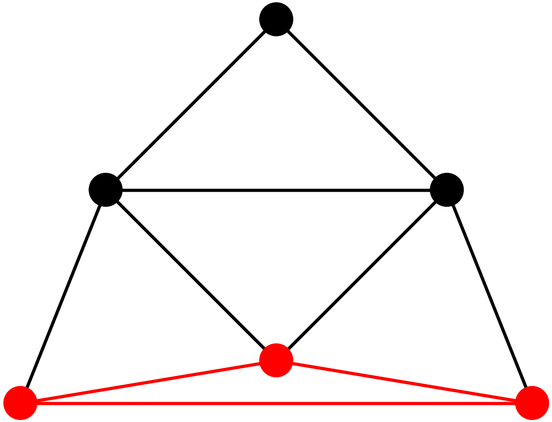

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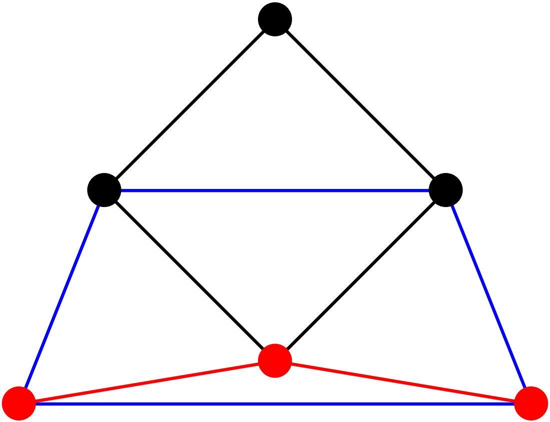

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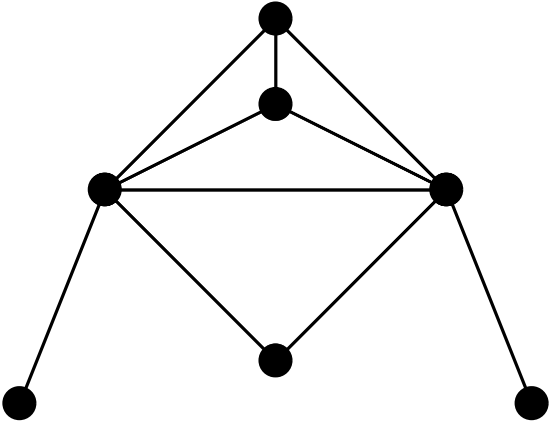

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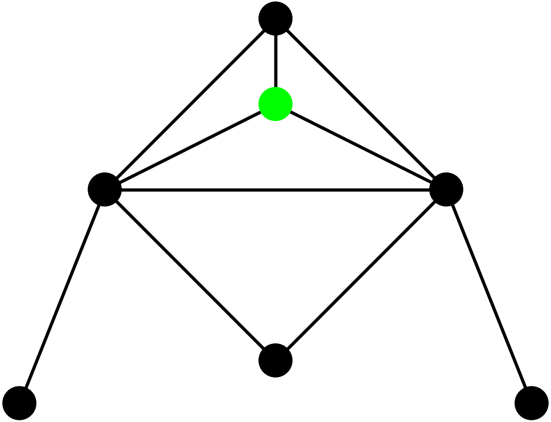

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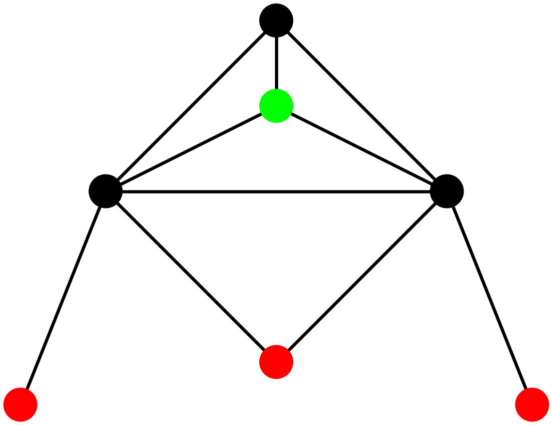

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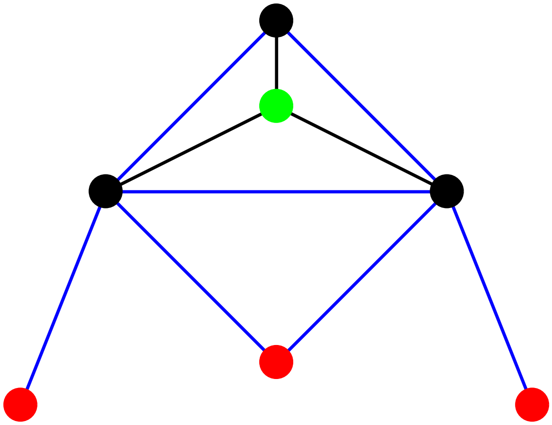

Probe-$\{3K_1, C_4, C_5\}$ graphs

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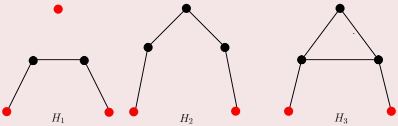
Remark

Given a graph G, then G^+ is probe unit interval if and only if G is probe-$\mathcal{R}$.

Partitioned probe-$\mathcal{R}$ graphs

Lemma

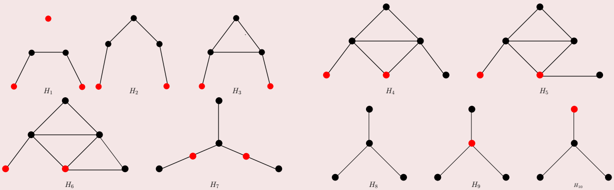
Given a partitioned interval graph $G = (P \cup N, E)$, then $G = (P \cup N, E)$ is a partitioned probe-$\mathcal{R}$ graph if and only if every three pairwise nonadjacent vertices has at most one vertex in P and it does not contain any of the following graph as partitioned induced subgraph.



Partitioned probe unit interval graphs

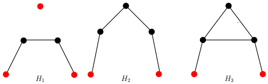
Theorem

Given a partitioned interval graph G, then G is a partitioned probe unit interval graph if and only if G contains no H_i^+ for $1 \leq i \leq 3$ and H_j for $4 \leq j \leq 10$ as partitioned induced subgraphs.



Sketch of the proof

- Suppose first that G has no universal vertices. Let $\mathcal{I}$ be an interval model of G. If G has no of these graphs, plus a universal vertex, as partitioned induced subgraph, then G is a partitioned probe unit interval graphs.



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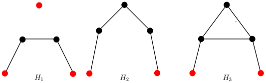

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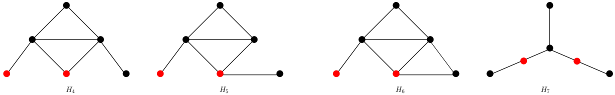

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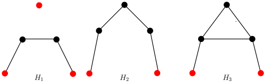
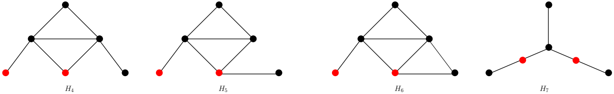
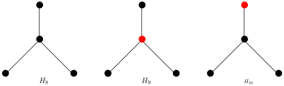


- Suppose now that G has no universal vertex. If G has no of these graphs as partitioned induced subgraphs, then G has an interval model $\mathcal{I}'$ with no centered probe vertex.



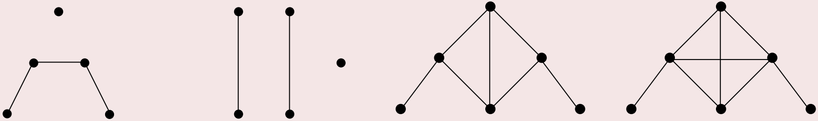
- If G has no of these graphs as partitioned induced subgraph, then we can build from $\mathcal{I}'$ a unit interval completion for G.



Unpartitioned probe-$\mathcal{R}$-graph

Lemma

Given an interval graph G, then G is a probe-$\mathcal{R}$ graph if and only if G does not contain any of the following graphs as forbidden induced subgraphs.

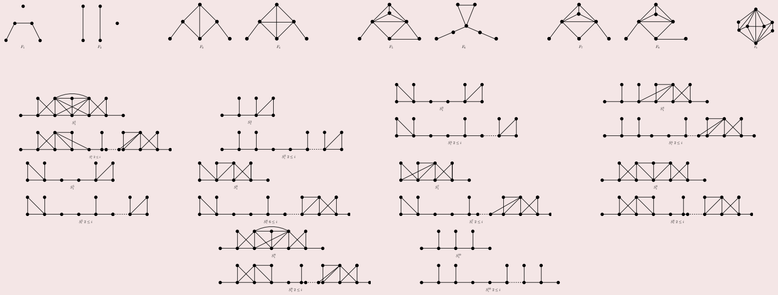


F_1 F_2 F_3 F_4

Unpartitioned probe unit interval graphs

Theorem

Given an interval graph G, then G is probe unit interval if and only if it contains no F_i^+ for $1 \leq i \leq 4$, F_i for $5 \leq i \leq 9$ and S_i^j for $1 \leq i$ and $1 \leq j \leq 10$.



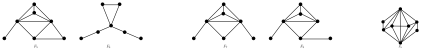
Sketch of the proof

- Suppose first that G has a universal vertex. Let $\mathcal{I}$ be an interval model of G. If G has no of these graphs, plus a universal vertex, as induced subgraphs, then G is a partitioned probe unit interval graph.

Sketch of the proof

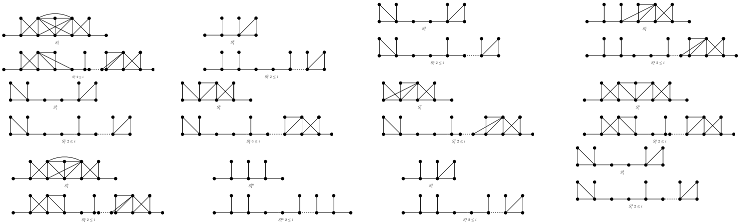
- Suppose first that G has a universal vertex. Let $\mathcal{I}$ be an interval model of G. If G has no of these graphs, plus a universal vertex, as induced subgraphs, then G is a partitioned probe unit interval graph.

- Suppose now that G has no universal vertex. If G has no of these graphs as induced subgraphs, then G has an interval model $\mathcal{I}'$ with no two pairwise adjacent centered vertices.



Sketch of the proof

- If G has no of these infinite family of graphs, then we can build from $\mathcal{I}'$ a probe unit interval partition for G.



Thank you for your attention!