Characterization by forbidden induced graphs of some subclasses of chordal graphs

Sérgio H. Nogueira[1,2] e Vinicius F. dos Santos[1,3]

[1] PPGMMC, CEFET-MG
[2] Instituto de Ciências Exatas e Tecnológicas, UFV
[3] Departamento de Ciência da Computação, UFMG

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Definitions

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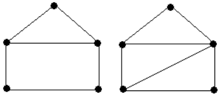
A graph is chordal if every cycle of length greater than three has a chord.*

*An edge connecting two nonconsecutive vertices.



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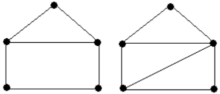

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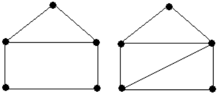

- Minimal separators

Definition

A set $S \subset V(G)$ disconnects a vertex a from b in G if every path of G between a and b contains a vertex from S. A non-empty set $S \subset V(G)$ is a minimal separator of G if there exist a and b such that

S disconnects a from b in G and

no proper subset of S disconnects a from b in G.

- Clique tree

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Definition

A clique tree of a connected chordal graph is any tree T whose vertices are the maximal cliques of G such that for every two cliques C_1, C_2 each clique on the path from C_1 to C_2 in T contains $C_1 \cap C_2$.

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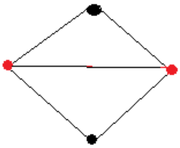
- Separating pair

Definition

Two maximal cliques C_1, C_2 of G form a separating pair if $C_1 \cap C_2$ is non-empty, and every path in G from a vertex of $C_1 \backslash C_2$ to a vertex of $C_2 \backslash C_1$ contains a vertex of $C_1 \cap C_2$.



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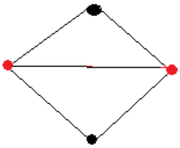

Main results

Theorem

A graph G is chordal if and only if every minimal separator of G is a clique. (Golumbic, 2004) [4]

Theorem

Let G be a chordal graph. The multiset $\mathbf{S}$ of minimal separators of vertices of G is the same for every clique tree T of G.(Blair and Peyton, 1992) [1]

Theorem

A set S is a minimal separator of a chordal graph G if and only if there exist maximal cliques C_1, C_2 forming a separating pair such that $S = C_1 \cap C_2$. (Habib and Stacho, 2012) [5]

Goal

Our goal is to characterize a subclass of chordal graphs by the intersection of minimal separators and forbidden induced subgraphs.

Theorem

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Let G be a chordal graph and let $\mathbf{S} = \{S_1, S_2, , ..., S_n\}$ the multiset of minimal separators of G. Then:

- *(i) For every $S_i, S_j \in \mathbf{S}, i \neq j, S_i \cap S_j = \emptyset \Leftrightarrow G$ is (claw, gem)-free.*

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  .

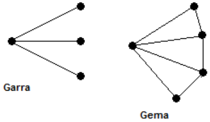
  Other results:

  .

- *(ii) (For every $S_i, S_j \in \mathbf{S}, S_i \cap S_j \neq \emptyset \Rightarrow S_i = S_j$) $\Leftrightarrow$ G is (dart, gem)-free. (strictly chordal graphs) Markezon and Waga, 2015) [6], [7].*
- *(iii) (For every $S_i, S_j \in \mathbf{S}, i \neq j, S_i \cap S_j \neq \emptyset \Rightarrow S_i \subsetneq S_j$ or $S_j \subsetneq S_i$) $\Leftrightarrow G$ is gem-free.*
- *(iv) Π_4: For every $S_i, S_j \in \mathbf{S}, i \neq j, S_i \cap S_j \neq \emptyset \Rightarrow S_i \nsubseteq S_j$ and $S_j \nsubseteq S_i$. A chordal graph is Π_4 hereditary $\Leftrightarrow G$ is dart-free. (($\Leftarrow$) De Caria and Gutiérrez, 2016) [3]*

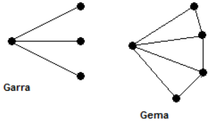
Proof

$S_i \cap S_j = \emptyset \Leftrightarrow G$ is (claw, gem)-free.

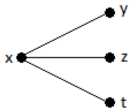


Proof

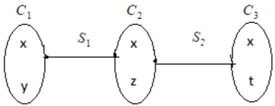
$S_i \cap S_j = \emptyset \Leftrightarrow G$ is (claw, gem)-free.



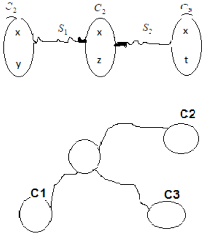
($\Rightarrow$) Suppose that G has a claw or a gem.
If it has a claw,



let x, y, z, t be the vertices of the claw and let C_1, C_2, C_3 cliques containing $\{x, y\}, \{x, z\}, \{x, t\}$, respectively.

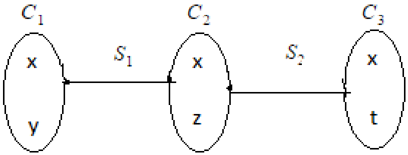


We can have other situations, as



But it is analogous.

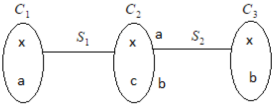
Without loss of generality, we may consider only the case



Then $x \in S_1 = C_1 \cap C_2$ e $x \in S_2 = C_2 \cap C_3$. Then $x \in S_1 \cap S_2 \neq \emptyset$. If G has a gem, analogously let C_1, C_2, C_3 be cliques containing $\{x, y, z\}, \{x, z, t\}, \{x, t, w\}$, respectively. Then the minimal separators are $S_1 = C_1 \cap C_2 \supseteq \{xz\}$, $S_2 = C_2 \cap C_3 \supseteq \{xt\}$. Therefore $x \in S_1 \cap S_2 \neq \emptyset$.

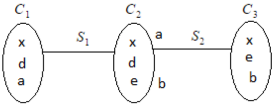
($\Leftarrow$) Conversely, suppose $S_1 \cap S_2 \neq \emptyset$. We may assume that S_1 and S_2 are adjacent (if this is not the case, since the intersection is no empty, there exist adjacent cliques that give us the same result). In this case let C_1, C_2, C_3 be cliques such that $S_1 = C_1 \cap C_2$, $S_2 = C_2 \cap C_3$ e $x \in S_1 \cap S_2$, i.e., $x \in C_1 \cap C_2 \cap C_3$. Since the cliques are distinct and maximal then
$\exists a \in C_1 \setminus C_2 \neq \emptyset$
$\exists b \in C_3 \setminus C_2 \neq \emptyset$
$C_2 \setminus C_1 \neq \emptyset$
$C_2 \setminus C_3 \neq \emptyset$
We have two cases:

Case 1 : $\exists c \in C_2 \setminus (C_1 \cup C_3) \neq \emptyset$

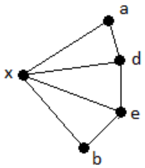


In this case we have a claw.

Case 2: $C_2 \setminus (C_1 \cup C_3) = \emptyset$.
$\exists e \in C_2 \setminus C_1 \subset C_3$
$\exists d \in C_2 \setminus C_3 \subset C_1$



In this case we have a gem.



Future researchs

- combination of intersections

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- combination of intersections
- minimal separators and Helly property

References

- Blair, J.R.S., Peyton, B., An Introduction to Chordal Graphs an Clique Trees. *Graph Theory and Sparc Matrix Computation*, IMA 56, 1992 1-29.
- De Caria, P., Gutiérrez, M., Introducing subclasses of basic chordal graphs, *Eletronic Notes in Discrete Mathematics*, 44, 2014, 287-292.
- De Caria, P., Gutiérrez, M., On basic chordal graphs and some of its subclasses, *Discrete Applied Mahematics*, 210, 2016, 261-276.
- Golumbic, M. C. *Algorithmic Graph Theory and Perfect Graphs (2nd edition)*, North Holland, (2004).
- Habib, M., Stacho, L., Reduced clique graphs of chordal graphs, *European Journal of Combinatorics*, 33, (2012), 712-735.
- Markenzon, L., Waga, C.F.E.M.,Strictly Cordal Graphs:Characterization and Linear Time Recongnition, *Eletronic Notes in Discrete Mathematics*, 52, (2016),181-188.
- Markenzon, L., Waga, C.F.E.M., New results on ptolemaic graphs, *Discrete Applied Mathematics*, 195,(2015),135-140.