Biclique graph of a subclass of split graphs

Marina Groshaus[1 2] André Guedes [3] Juan Pablo Puppo [4]

[2]Universidad de Buenos Aires, Argentina

[3]Universidade Federal do Paraná, Brazil

[4]Universidad Tecnológica Nacional, Argentina

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Outline

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1. Preliminaries
2. Our work

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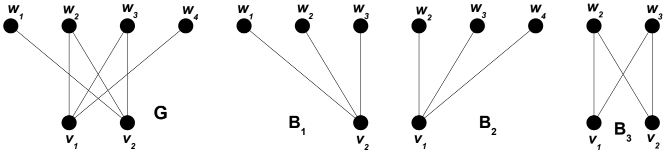
Bicliques

Definition

A *biclique* of a graph G is a maximal bipartite complete subgraph



Biclique Graph

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Given a graph G, the biclique graph $KB(G)$ is the intersection graph of bicliques of a graph. The biclique graph was introduced by Groshaus and Szwarcfiter [Groshaus and Szwarcfiter, 2010].

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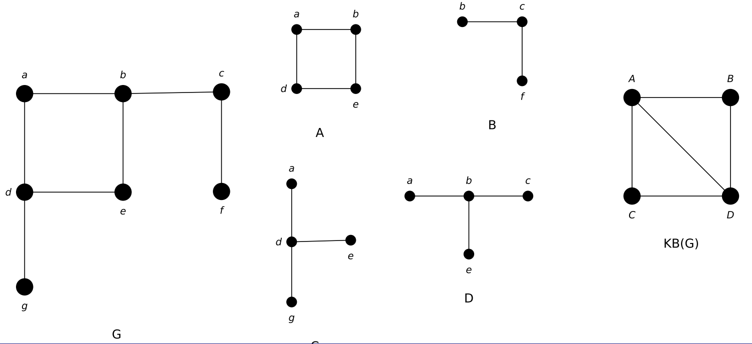

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- Since then, it remains open the time complexity of the problem of recognizing biclique graphs.

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The biclique graph can be considered as an operator between graphs:
Given a graph H, the operator KB returns another graph, the biclique graph of G, $KB(G)$.

If H is such that $KB(H) = G$, we say that H is the *pre-image* of G.

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  - Recognizing the class $\mathcal{B}$ such that $\mathcal{H}(\mathcal{B}) \subseteq \mathcal{A}$. (Find the time complexity of the problem of deciding if a given graph belongs to $\mathcal{H}^{-1}(\mathcal{A})$).

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Some examples for the Clique graph operator

In the context of the clique graph operator (intersection graphs of the cliques of a graph). Some examples [Szwarcfiter, 2003].

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Also, the clique graph of the split, diamond-free, dismantable graphs, arc-circular graphs etc were studied.

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We solve the problem for a subclass of split graphs, called the nested separable split graphs.

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Split graph $H = (K \cup S, E)$: vertices can be partitioned into a clique K and an independent set (*satellites*).

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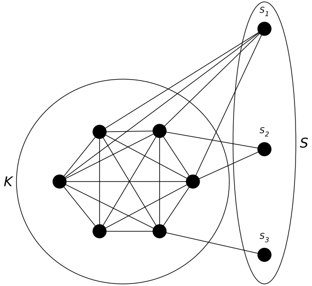

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- Note that $(\mathcal{X}, \preceq)$ is a transitive, anti-symmetric and reflexive relation, that is, is a poset.

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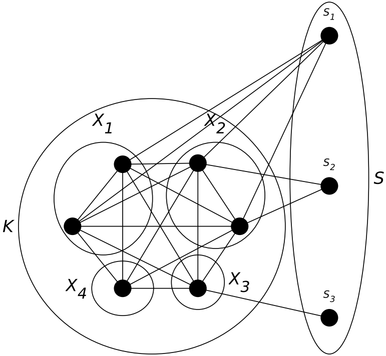

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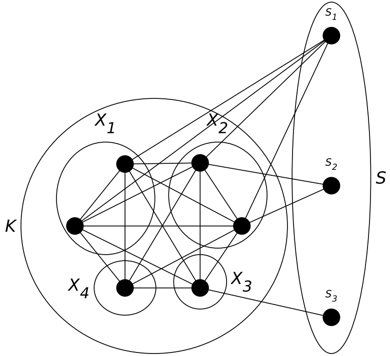

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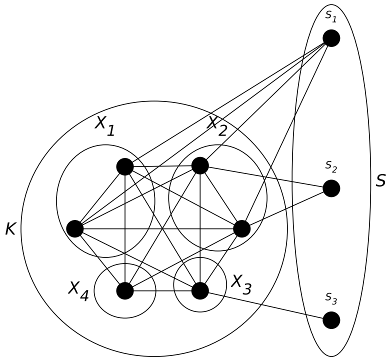

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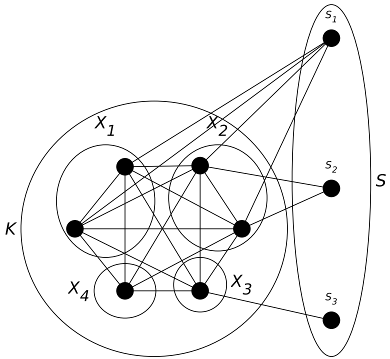

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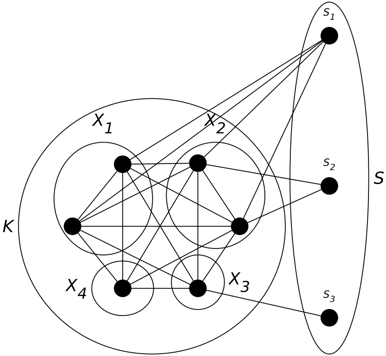

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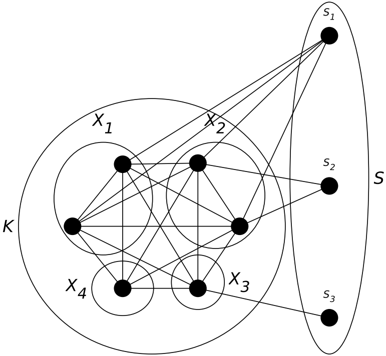

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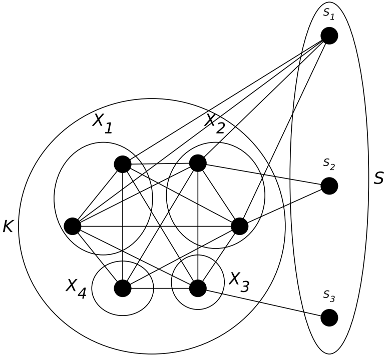

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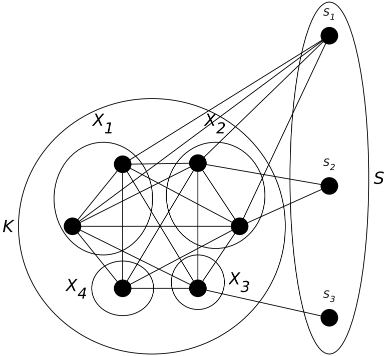

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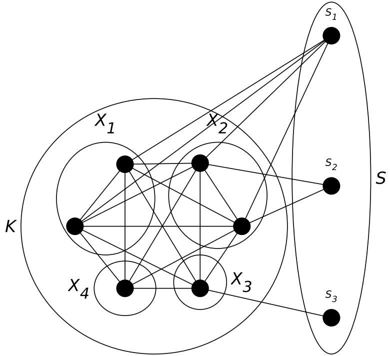

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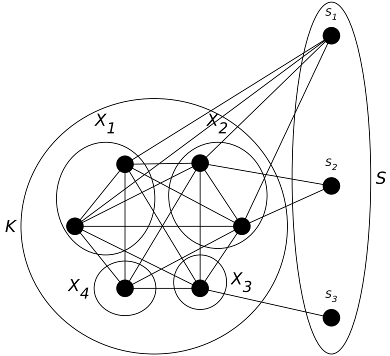

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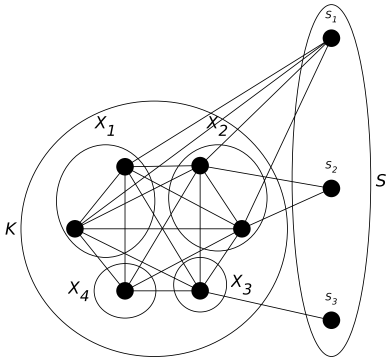

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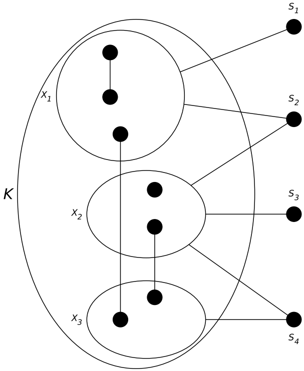
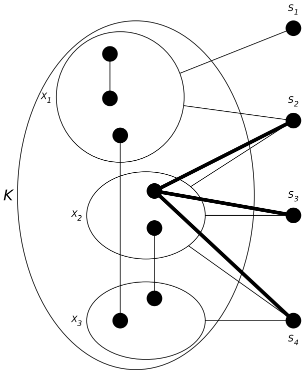
  Note that there are two different bicliques for each pair of vertices u,v.

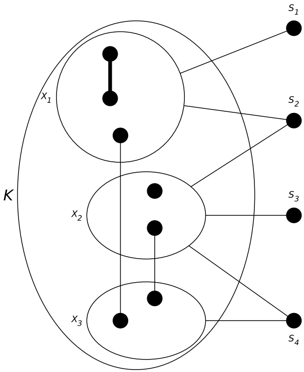
bicliques of splits graphs



bicliques of splits graphs - star-biclique



bicliques of splits graphs - edge-biclique



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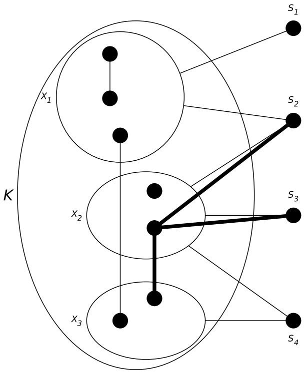

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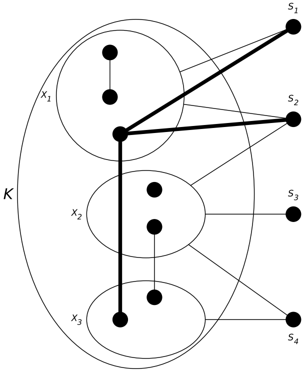

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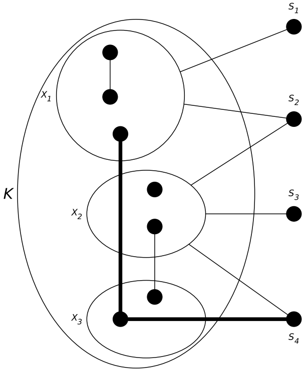

Degree of vertices of $KB(H)$

Lemma

Let H be a split graph that does not contain star bicliques and that every part contains at least 3 vertices. If $d_{KB(H)}(v) = \delta(KB(H))$, then v is an edge-biclique of some part X_i of H.

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- A separable split graph does not have star-bicliques.

Bicliques of H and edges of K

Consider a nested separable split graph H.

- Each biclique B of H contains exactly one edge of K. Call that edge the *base edge* of B.
- Each edge of K belongs to one or two bicliques of H.

Cliques of $KB(H)$

Definition

A *star-clique* C^v of G is a clique with all the bicliques of H that contain vertex $v \in K$. That is, the base edges of the bicliques in C^v form a star subgraph in K.

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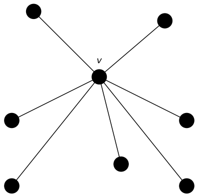
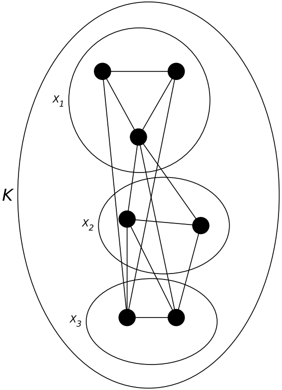


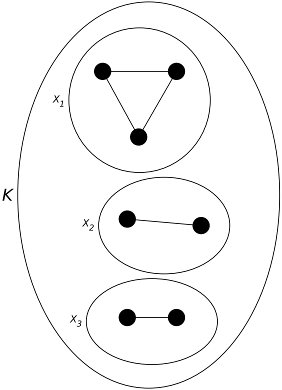
$KB(H)$

Observation

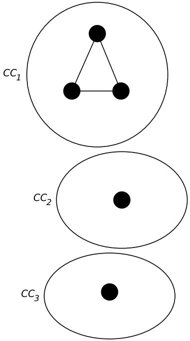
Let $\mathcal{E}$ be the set of edge-bicliques of H.
Each connected component, CC_i of the $G[\mathcal{E}]$ is associated to a part X_i of the partition $\mathcal{X} = \{X_1, \ldots, X_\ell\}$ of K in H.
The vertices (edge-bicliques of H) of the connected component CC_i of $G[\mathcal{E}]$ are the edges of the graph $H[X_i]$ (subgraph of H induced by X_i) which is a complete graph, and $|CC_i| = \binom{|X_i|}{2}$.



Graph H



Internal edges



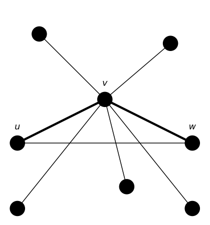
Connected components of G[ε]

Cliques of $KB(H)$

Observation (separable)

Let B_{uv} and B_{vw} be two adjacent edge-bicliques in G.
The intersection of the closed neighbourhood in G of B_{uv} and B_{vw} is formed by the star-clique C^v and the edge-biclique B_{uw}.

That is, $N_G[B_{uv}] \cap N_G[B_{vw}] = C^v \cup \{B_{uw}\}$.



Observe that the biclique B_{uw} is adjacent only to B_{uv} and B_{vw} when restricted to $N_G[B_{uv}] \cap N_G[B_{vw}]$.

Main problem

Problem

Given a graph G, decide if G is a biclique graph of a nested separable split graph.

In the case the answer is YES, we want to find the pre-image of G.

Skecht of the Algorithm

We propose an algorithm that builds a candidate H such that if G is $KB(F)$ for some nested separable split graph F then $G \simeq KB(H)$.

Skecht of the Algorithm

The idea of the algorithm is the following:

- First we discover which vertices are the only possible vertices of G that are candidates to represent the edge-bicliques of a pre-image. (Using minimun degree)
- We look in G for the candidates for the star-cliques. (using intersection of neghborhoods) and use them to find the candidate edge of K of each biclique.
- We find the candidate order for the parts using intersection of the bicliques.
- We construct the corresponding set of satellites and the candidate pre-image graph H such that, if G is a biclique graph for some separable split graph, then $G \simeq KB(H)$.

Algorithm

Given a graph $G = (V, E)$, the steps of an algorithm are:

1. Find the edge-bicliques candidates $\mathcal{E}$;
2. Find the connected components of $G[\mathcal{E}]$, $CC_1, \ldots, CC_\ell$ and generate the candidate partition $\mathcal{X}$ and discover the size n_i of each part.
3. Find the candidates star-cliques
4. Mark the single s-bicliques and the double s-bicliques;
5. Find the associated poset $(X, \preceq)$
6. Generate the sets $S(X_1), \ldots, S(X_\ell)$, construct a candidate nested separable split graph H and check if $G \simeq KB(H)$.

Groshaus, M. and Szwarcfiter, J. L. (2010).
Biclique graphs and biclique matrices.
Journal of Graph Theory 63 (1), 1-16.

Szwarcfiter, J. (2003).
A survay on clique graphs.
Recent Advances in Algorithms and Combinatorics CMS Books in Mathematics, 109-136.