Equitable total colouring of Loupekine Snarks and its products

Luana Cordeiro (UFF, Rio de Janeiro, Brazil)

Simone Dantas (UFF, Rio de Janeiro, Brazil)

Diana Sasaki (UERJ, Rio de Janeiro, Brazil)

November, 2016

Edge Colouring

Definition

A k-**edge colouring** of a graph G is an assignment of k colours to the edges of G in such a way that two adjacent edges have different colours.

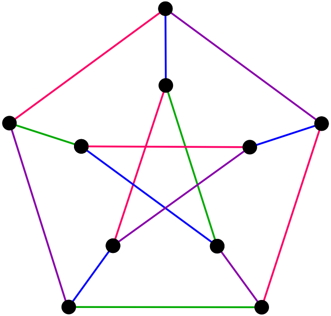


Figure: A 4-edge colouring of the Petersen graph.

Edge Colouring

Theorem (Vizing, 1964)
Every simple graph admits an edge colouring with at least Δ and at most $\Delta + 1$ colours.

The **chromatic index** of G — $\chi'(G)$ — is the least k for which G admits a k-edge colouring.

For cubic graphs, those with $\chi' = 3$ are said to be **Class 1**, and those with $\chi' = 4$ are said to be **Class 2**.

Total Colouring

A k-**total colouring** of a graph G is an assignment of k colours to its vertices and edges such that two adjacent or incident elements have different colours.

The **total chromatic number** of G — $\chi''(G)$ — is the least k for which G admits a k-total colouring.

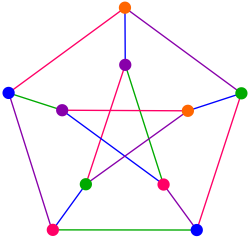


Figure: A total colouring of the Petersen graph.

Total Colouring

Total Colouring Conjecture (Behzad, 1964 - Vizing, 1967)
Every simple graph admits a total colouring using at least $\Delta + 1$ and at most $\Delta + 2$ colours.

It was proved for **cubic graphs** in 1971 independently by Rosenfeld and Vijayaditya.

Cubic graphs with $\chi'' = 4$ are said to be **Type 1** and cubic graphs with $\chi'' = 5$ are said to be **Type 2**.

Equitable Total Colouring

An **equitable k-total colouring** of a graph is a k-total colouring of its elements such that the cardinalities of any two colour classes differ by at most 1.

The **equitable total chromatic number** of G — $\chi''_e(G)$ — is the least k for which G admits an equitable k-total colouring.

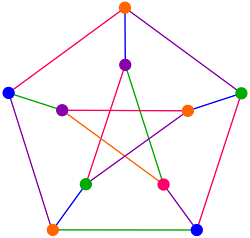


Figure: An equitable 5-total colouring of the Petersen graph.

Equitable Total Colouring

Equitable Total Colouring Conjecture (Wang, 2002)
Every simple graph admits an equitable total colouring with at least $\Delta + 1$ and most $\Delta + 2$ colours.

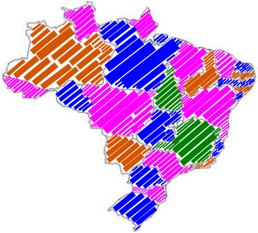
Wang proved that it holds for cubic graphs.

Sasaki et al. proved that the problem of deciding whether the equitable total chromatic number of a cubic graph is 4 or 5 is NP-complete.

Snarks

The Four Colour Conjecture (Francis Guthrie, 1852)

No more than four colours are required to colour the regions of a map so that no two adjacent regions have the same colour.



First proved in 1976 by Appel and Haken.

Theorem (Tait, 1880)

The Four Colour Conjecture is equivalent to the statement that every cubic bridgeless planar graph is Class 1.

Snarks were defined in this context, trying to find a counterexample to the Four Colour Conjecture (a cubic bridgeless planar graph of Class 2).

Snarks

Snarks are bridgeless cubic graphs of Class 2.

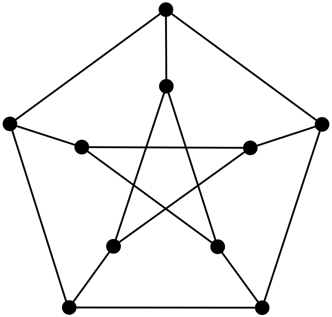


Figure: The Petersen graph is the smallest snark.

Snarks

- In 2003, Cavicchioli et al. verified that all snarks with girth at least 5 and fewer than **30** vertices are Type 1;
- In 2011, Brinkmann et al. verified that all snarks with such girth and fewer than **38** vertices are Type 1;
- In 2011, Campos et al. proved that the infinite families of Flower and Goldberg snarks are Type 1. Moreover, they have equitable total chromatic number 4;
- Later on, Sasaki et al. proved that both Blanuša families and a part of Loupekine family are Type 1, but they could not determine equitable total colourings of Loupekine snarks using 4 colours.

Snarks

The importance of these graphs arise from the fact that snarks are counterexamples for many conjectures in Graph Theory.

In this work, we present an equitable total colouring with 4 colours for an infinite family of Loupekine snarks and also for some of its *dot products* with Flower and Blanuša snarks.

Loupekine Snarks

Snark L_0 is presented on the left. The next ones are obtained by deleting the dashed edges and adding a copy of the block B depicted on the right.

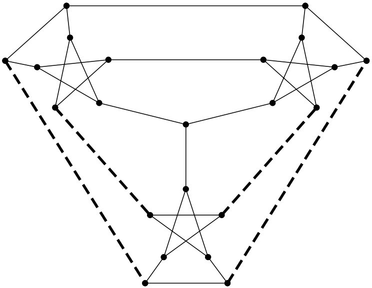


Figure: The first snark L_0.

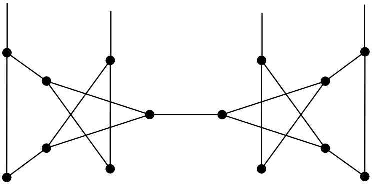


Figure: The block B.

Loupekine Snarks

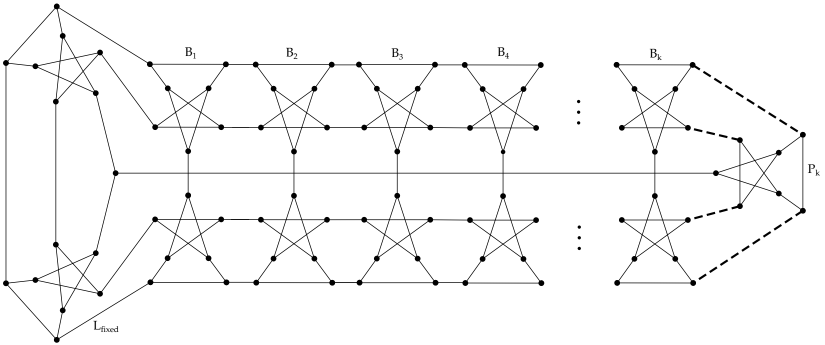


Figure: L_k, obtained from L_0 by adding k blocks.

Loupekine Snarks

In 2014, Sasaki et al. proved that all members of a part of the Loupekine family is Type 1, but the colourings are not equitable.

Theorem

All members of this Loupekine subfamily have equitable total chromatic number equal to 4.

The proof is by construction and four different equitable 4-total colourings were necessary to obtain the result.

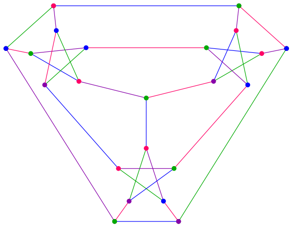


Figure: An equitable 4-total colouring of L_0.

Loupekine Snarks

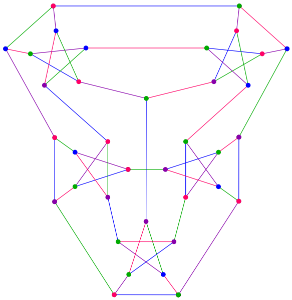


Figure: An equitable 4-total colouring for L_1

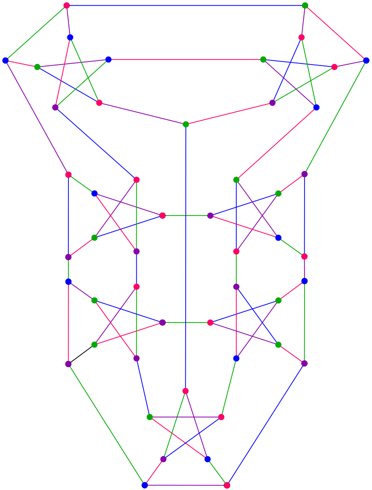


Figure: An equitable 4-total colouring for L_2

Loupekine Snarks

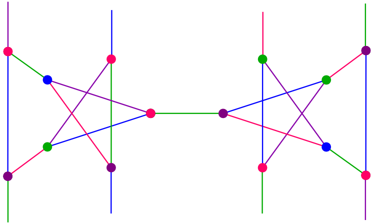


Figure: B_k for $k \equiv 1 \mod 4$

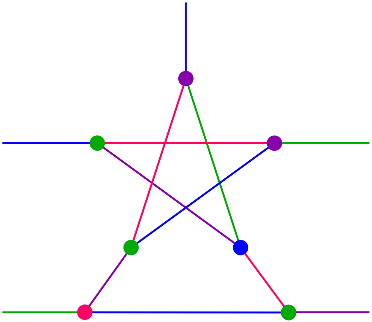


Figure: P_k for $k \equiv 1 \mod 4$

Loupekine Snarks

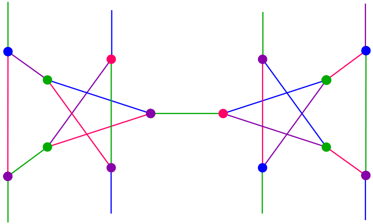


Figure: B_k for $k \equiv 2 \mod 4$

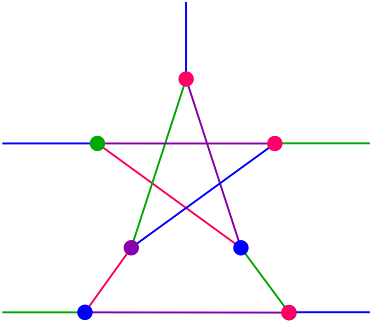


Figure: P_k for $k \equiv 2 \mod 4$

Loupekine Snarks

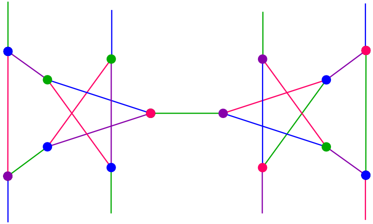


Figure: B_k for $k \equiv 3 \mod 4$

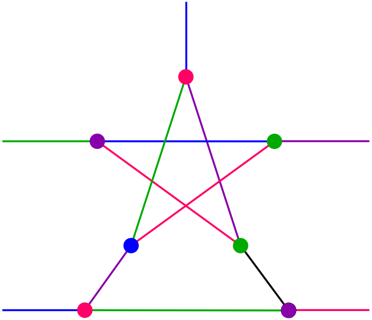


Figure: P_k for $k \equiv 3 \mod 4$

Loupekine Snarks

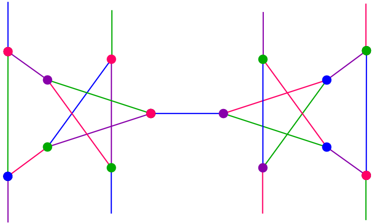


Figure: B_k for $k \equiv 0 \mod 4$

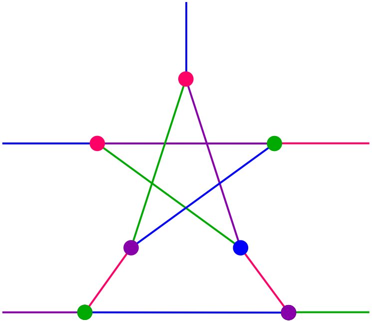


Figure: P_k for $k \equiv 0 \mod 4$

Loupekine Snarks

We represent 1 for pink, 2 for green, 3 for blue and 4 for purple.
$\varphi(k)$ denotes the number of elements coloured with colour k.

For L_0, we have $\varphi(1) = 14$, $\varphi(2) = 14$, $\varphi(3) = 14$ and $\varphi(4) = 13$.
For L_k, by using B_k and P_k, $k \mod 4$, we obtain the following cardinalities of the first members.

	$\varphi(1)$	$\varphi(2)$	$\varphi(3)$	$\varphi(4)$
$B_1 + P1$	$+9$	$+9$	$+8$	$+9$
$B_1 + B_2 + P_2$	$+18$	$+17$	$+17$	$+18$
$B_1 + B_2 + B_3 + P_3$	$+26$	$+26$	$+26$	$+27$
$B_1 + B_2 + B_3 + B_4 + P_4$	$+35$	$+35$	$+35$	$+35$
$B_1 + B_2 + B_3 + B_4 + B_1 + P_1$	$+44$	$+44$	$+44$	$+43$

Loupekine Snarks

Therefore, for snark L_k, the number of elements with each colour is
$\varphi(1) = n$, $\varphi(2) = n$, $\varphi(3) = n$ and $\varphi(4) = \mathbf{n\text{-}1}$
if $k \equiv 0 \mod 4$;

$\varphi(1) = n$, $\varphi(2) = n$, $\varphi(3) = \mathbf{n\text{-}1}$ and $\varphi(4) = \mathbf{n\text{-}1}$
if $k \equiv 1 \mod 4$;

$\varphi(1) = n$, $\varphi(2) = \mathbf{n\text{-}1}$, $\varphi(3) = \mathbf{n\text{-}1}$ and $\varphi(4) = \mathbf{n\text{-}1}$
if $k \equiv 2 \mod 4$

$\varphi(1) = n$, $\varphi(2) = n$, $\varphi(3) = n$ and $\varphi(4) = n$
if $k \equiv 3 \mod 4$.

Dot Product

A **dot product** of two cubic graphs is a cubic graph obtained by deleting two nonadjacent edges of one graph and two adjacent vertices of the other graph, and then adding edges between the vertices of degree 2. A dot product of two snarks is a snark.

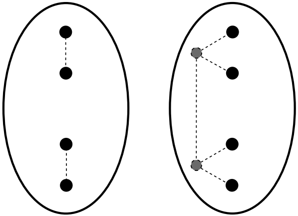


Figure: The dot product: relevant elements.

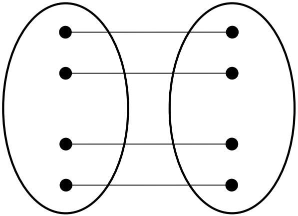


Figure: The resulting graph.

Flower Snarks

The first Flower snark is presented on the left. The next elements in the family are obtained by deleting the dashed edges and adding the block depicted on the right.

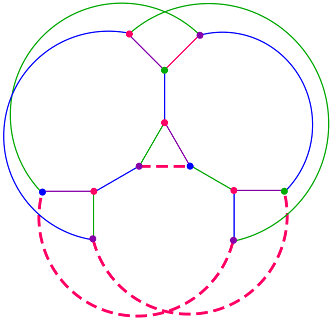


Figure: Graph F_3

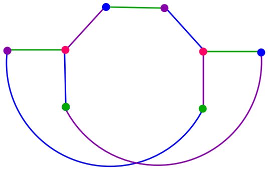


Figure: The block FL

Flower Snarks

An equitable 4-total colouring for all members of this family was determined by Campos et al. in 2011.

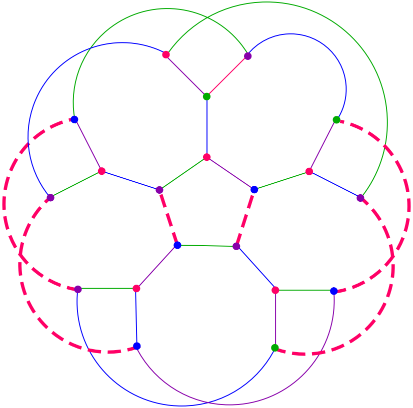
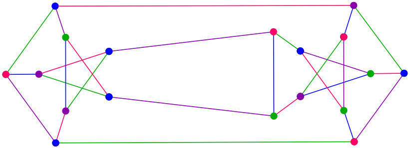
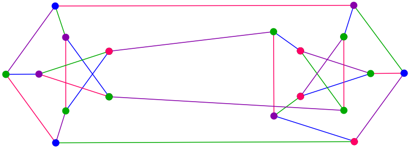


Figure: Graph F_5

Blanuša Snarks

Due to the high symmetry of the Petersen graph, there are only two Blanuša graphs.





Sasaki et al. determined equitable 4-total colourings for all members of Blanuša family.

Blanuša Snarks

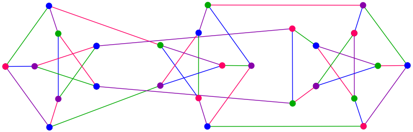


Figure: The second graph in the first Blanuša family.

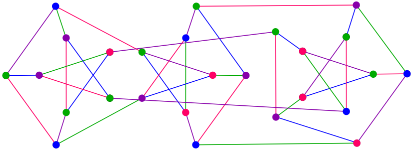
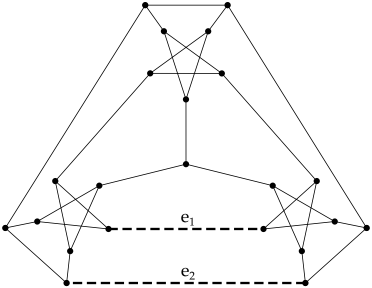


Figure: The second graph in the second Blanuša family.

Equitable Total Colouring of snark products

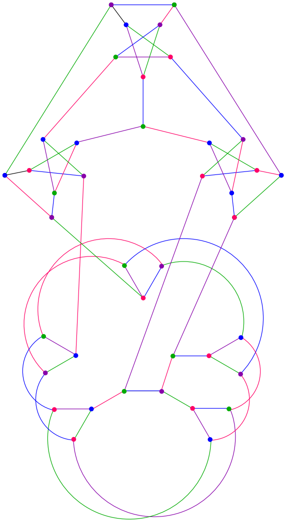
Theorem

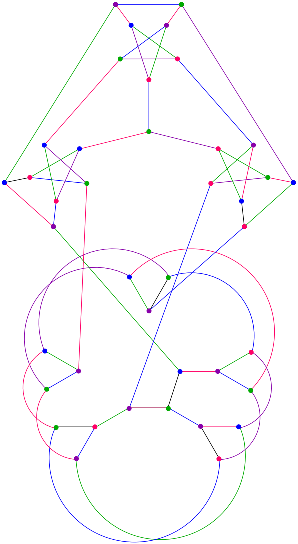
All graphs obtained by the dot product of a Loupekine snark using edges e_1 and e_2 and all Flower snarks have equitable total chromatic number 4.



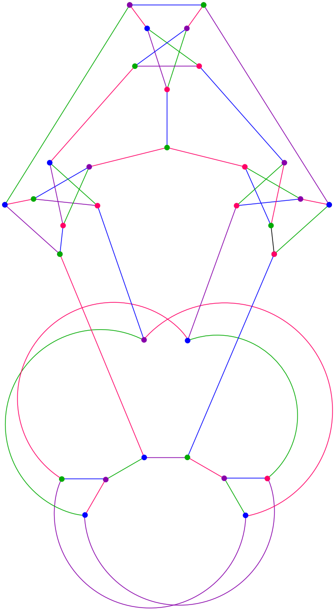
In the dot product of Loupekine snarks and others, we remove edges e_1 and e_2 of each Loupekine snark.

Examples

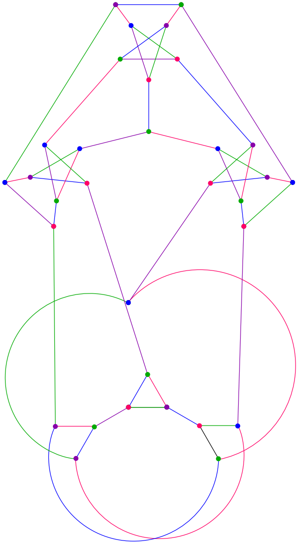


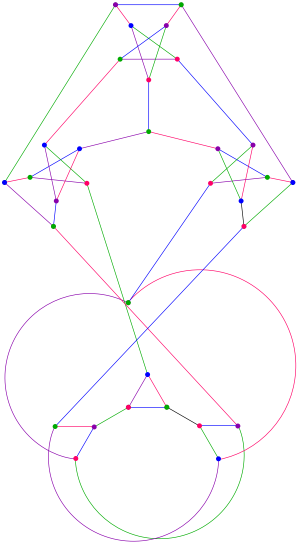


Examples

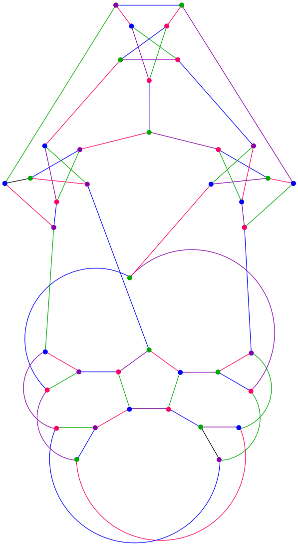


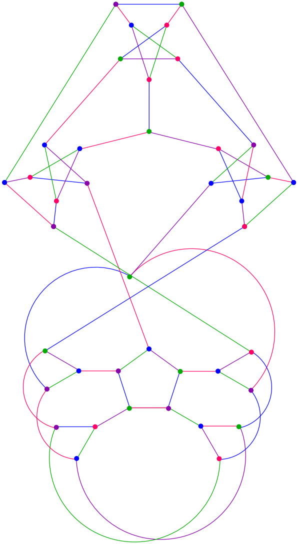
Examples





Examples



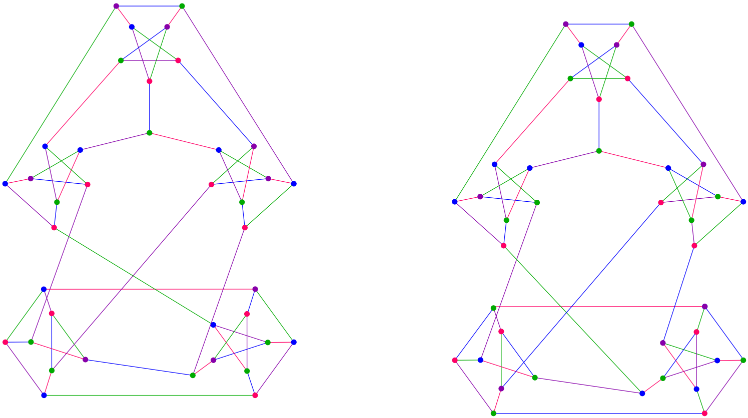


Equitable Total Colouring of snark products

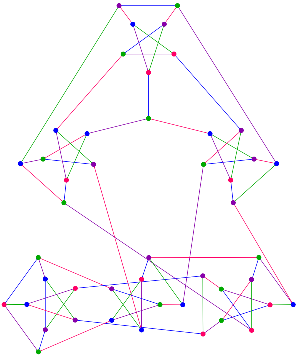
Theorem

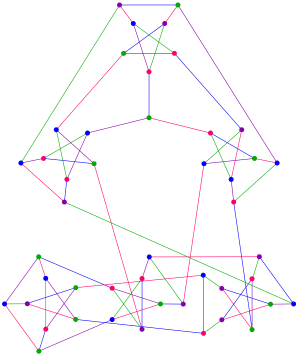
All graphs obtained by the dot product of a Loupekine snark using edges e_1 and e_2 and all Blanuša snarks have equitable total chromatic number 4.

Examples

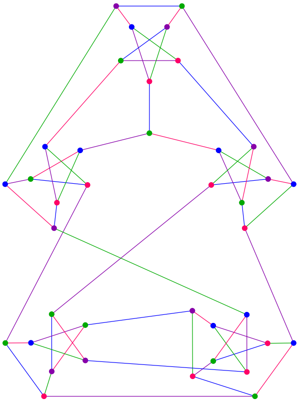


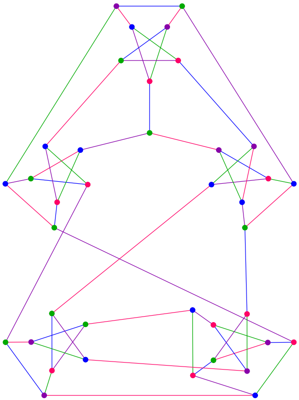
Examples





Examples





References

- Campos, C. N., Dantas, S., De Mello, C. P. The Total-chromatic Number of Some Families of Snarks, Discrete Mathematics 311, 2011, 984-988.
- Dantas, S., De Figueiredo, C. M. H., Mazzuoccolo, G., Preissmann, M., Dos Santos, V. F., Sasaki, D. On the equitable total chromatic number of cubic graphs. Discrete Applied Mathematics, 209, 84-91, 2016.
- Isaacs, R. Infinite families of nontrivial trivalent graphs which are not Tait colorable. Amer. Math. Monthly, v. 82, n. 3, 221–239, 1975.
- Preissmann, M. Snarks of order 18. Discrete Math., 42, 125-126, 1982.
- Sasaki, D. Coloração total de famílias de snarks. Master's thesis, Programa de Engenharia de Sistemas e Computação COPPE/UFRJ, 2010.
- Sasaki, D., Dantas, S., De Figueiredo, C. M. H., Preissmann, M. The hunting of a snark with total chromatic number 5. Discrete Applied Mathematics, 164, 470-481, 2014.
- Wang, W. F. Equitable total coloring of graphs with maximum degree 3, Graphs Combin 18, 2002, 677-685.