Characterization and linear-time detection of minimal obstructions to concave-round graphs and the circular-ones property

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Overview

Minimal obstructions to concave-round graphs

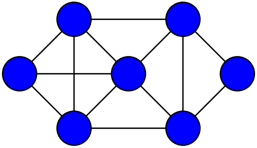
Minimal obstructions to the circular-ones property

Connections to other circular-arc graphs

Concave-round graphs

A graph is convave-round if there is a circular enumeration of its vertices in such a way that the closed neighborhood of each vertex is an interval in the enumeration.

Example:



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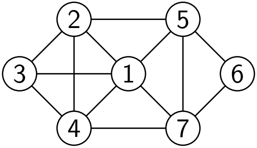

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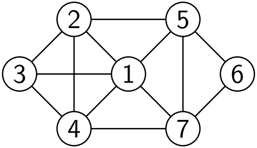
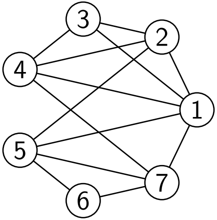


admits $1, 2, 3, 4, 5, 6, 7$ as a concave-round enumeration:



Concave-round graphs

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  In that work, they define convex-round graphs as those whose complement is concave-round.

Concave-round graphs

- Bang-Jensen, Huang, and Yeo (2000) proved that the chromatic number of concave-round graphs can be determined in $O(n^2)$ time, whereas graph coloring is NP-complete for general circular-arc graphs (Garey, Johnson, Miller, and Papadimitriou, 1980).

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- There are linear-time recognition algorithms for concave-round and convex-round graphs (Booth and Lueker, 1976).In linear time it is also possible to obtain negative certificates in the form of an odd cycle in an associated graph (McConnell, 2004; Kaplan and Nussbaum, 2009).

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- There are linear-time recognition algorithms for concave-round and convex-round graphs (Booth and Lueker, 1976).In linear time it is also possible to obtain negative certificates in the form of an odd cycle in an associated graph (McConnell, 2004; Kaplan and Nussbaum, 2009).
- Concave-round and convex-round graphs were also studied in connection to circular-perfectness (Bang-Jensen and Huang, 2002; Coulonges; 2006).

Concave-round graphs: minimal forbidden subgraphs

The main result of this work is the solution of the following problem:

Problem (Bang-Jensen, Huang, and Yeo, 2000)

Find a characterization by forbidden induced subgraphs of concave-round graphs (or, equivalently, for convex-round graphs).

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Moreover, we show that it is possible to find one of these forbidden induced subgraph in linear time in any graph that is not concave-round.

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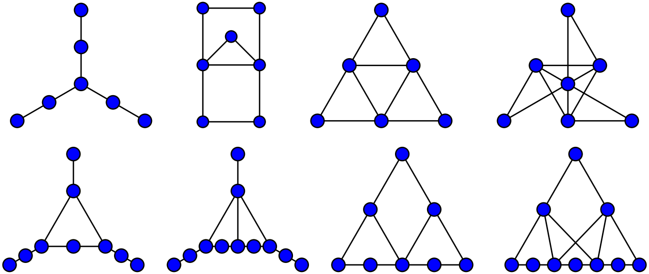
Theorem

A graph is concave-round if and only if G contains none of the following as induced subgraphs: $C_k \cup K_1$ for any $k \geqslant 4$, $\overline{C_{2k+1} \cup K_1}$ for any $k \geqslant 1$, $\overline{C_{2k}}$ for any $k \geqslant 3$, plus:



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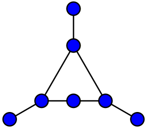
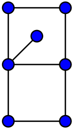
- As proper circular-arc graphs are concave-round (Tucker, 1971), G is not a proper circular-arc graph. Thus, G contains a forbidden induced subgraph for the class of proper circular-arc graphs. We are done unless $\overline{G}$ contains the complement of one of the following:

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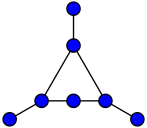
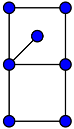



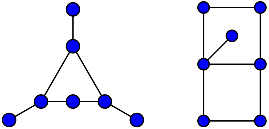
- Analyzing the different ways into which a chordless odd cycle may coexist in $\overline{G}$ with the two above graphs, we prove the presence of one of the forbidden induced subgraphs in our theorem. □

Concave-round graphs: forbidden subgraphs detection

The proof is constructive and has the following algorithmic consequence.

Corollary

Given a graph G, an odd chordless cycle C in $\overline{G}$ and an induced subgraph in $\overline{G}$ isomorphic to one of the following:



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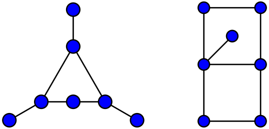

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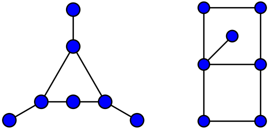

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- Our aim now is to show that it is possible to detect a minimal forbidden induced subgraph for the class of concave-round graphs in any given graph that is not concave-round.
- We achieve this by combining the above corollary with an algorithmic version of the characterization of PCA by Tucker (1974) due to Soulignac (2015), and algorithmic results that we obtain studying the minimal forbidden submatrices for the circular-ones property.

Consecutive-ones property

All the matrices in this work are binary.

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Consecutive-ones property (Fulkerson and Gross, 1964)

- A matrix has the consecutive-ones property (for rows) if by permuting its columns it is possible to obtain a matrix where the ones in each row form an interval.

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Containment as a configuration

Given two matrices M and M', we say that M contains M' as a configuration if M' equals some submatrix of M up to permutations of rows and of columns.

Consecutive-ones property: Tucker matrices

Theorem (Tucker, 1972)

A matrix has the consecutive-ones property if and only if it contains none of the following matrices as configurations:

$$M_I(k) = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad M_{II}(k) = \begin{pmatrix} 1 & 1 & & & & 0 \\ & 1 & 1 & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}$$

$$M_{III}(k) = \begin{pmatrix} 1 & 1 & & & & 0 \\ & 1 & 1 & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & 1 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}$$

$$M_{IV} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad M_V = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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Recently, it was proved that these matrices can be detected in linear time:

Theorem (Lindzey and McConnell, 2016)

Given a matrix M that does not have the consecutive-ones property, it is possible to find a Tucker matrix contained in M as a configuration in linear time, i.e., in $\mathrm{O}(\text{size}(M))$ time where

$$\text{size}(M) = \text{number of rows} + \text{number of columns} + \text{number of ones}.$$

Circular-ones property

Circular-ones property (Tucker, 1970)

- A matrix has the circular-ones property (for rows) if there is a circular ordering of its columns in such a way that that the ones in each row are consecutive in this circular ordering.

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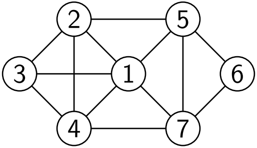
Concave-round graphs and the circular-ones property

An augmented adjacency matrix $M(G)$ of a graph G arises by putting 1's all along the diagonal of an adjacency matrix of G.

Concave-round graphs and the circular-ones property

A graph G is concave-round if and only if the augmented adjacency matrix $M(G)$ has the circular-ones property for rows and columns.



$$M(G) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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M^* and $\overline{M}$

Let M be a matrix.

- We denote by M^* the matrix that arises from M adding a last column consisting entirely of zeros;
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$a \oplus M$

If M is a matrix with k rows and $a = a_1 a_2 \dots a_k$ is a binary sequence of length k, we denote by $a \oplus M$ the matrix that arises by complementing the rows i of M such that $a_i = 1$.

$$1010 \oplus \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

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Shift and reversal of a sequence

If $a = a_1 a_2 \dots a_k$ is a binary sequence, we call the shift of a to the sequence $a_2 \dots a_k a_1$ and reversal of a to the sequence $a_k a_{k-1} \dots a_1$.

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Bracelet

A bracelet is the lexicographically smallest element in an equivalent class of binary sequences under shifts and reversals.

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A_k, $k \geqslant 3$

If $k \geqslant 4$, let A_k be the set of binary bracelets of length k.
If $k = 3$, let $A_3 = \{000, 111\}$. (The bracelets $001, 011 \notin A_3$.)

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In this work, we find such an analogous characterization.

Theorem

A matrix has the circular-ones property if and only if it contains no matrix in the following set as a configuration:

$$\mathcal{F}_{\text{circR}} = \{a \oplus M_I^*(k) : k \geqslant 3 \text{ and } a \in A_k\} \cup \{M_{IV}, \overline{M_{IV}}, M_V^*, \overline{M_V^*}\}.$$

Moreover, $\mathcal{F}_{\text{circR}}$ is a minimal such set and the matrices in the set $\mathcal{F}_{\text{circR}}$ are minimal forbidden submatrices for the circular-ones property.

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The proof relies on Tucker reductions and the characterization of the consecutive-ones property by forbidden submatrices by Tucker (1972).

Circular-ones property: minimal forbidden submatrices

Our characterization states that, apart from 4 sporadic matrices, the minimal forbidden submatrices for the circular-ones property, up to permutations of rows and of columns, are those obtained from a single family by row complementation:

$$M_I^*(k) = \begin{pmatrix} 1 & 1 & & & & 0 \\ & 1 & 1 & & & 0 \\ & & \ddots & \ddots & & 0 \\ & & & 1 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \quad \text{for } k \geqslant 3$$

Circular-ones property: minimal forbidden submatrices

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Another consequence is that the algorithm by Lindzey and McConnell (2016) to detect minimal forbidden submatrices for the consecutive-ones property can be used to detect the minimal forbidden submatrices for the circular-ones property.

Corollary

Given a matrix M that does not have the circular-ones property, it is possible to find in linear time a minimal forbidden submatrix for the circular-ones property contained in M as a configuration.

Circular-ones property: minimal forbidden submatrices

Our characterization also gives the number of minimal forbidden submatrices for the circular-ones property having k rows, up to permutations of rows and of columns. For $k = 3, 4, 5, 6, 7, 8 \ldots$, this number is $2, 10, 8, 13, 18, 30, \ldots$ and, for each $k \geqslant 5$, this number coincides with the number of binary bracelets of length k, which is known to be:

$$\frac{1}{2k} \sum_{d|k} \varphi(d) 2^{k/d} + \begin{cases} \frac{3}{4} \cdot 2^{k/2} & \text{if } k \text{ is even,} \\ \frac{1}{2} \cdot 2^{(k-1)/2} & \text{if } k \text{ is odd,} \end{cases}$$

where $d \mid k$ stands for 'd is a positive divisor of k' and φ is Euler's totient function.

Circular-ones property for rows and columns

From our results the circular-ones property for rows, we obtain analogous results for the circular-ones property for rows and columns. Let

$$\mathcal{A}_{\mathsf{circRC}} = \{0001, 0011, 0111, 00001, 00011, 00111, 01111, 000111\}$$

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Theorem

A matrix has the circular-ones property for rows and columns if and only if it contains neither D nor D^t as a configuration for any D in the set

$$\mathcal{F}_{\text{circRC}} = \bigcup_{k=3}^{\infty} \{M_I^*(k), \overline{M_I^*(k)}\} \cup \{a \oplus M_I^*(|a|) \colon a \in \mathcal{A}_{\text{circRC}}\} \cup \{M_V^*, \overline{M_V^*}\}$$

Moreover,

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Moreover,

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- and if M does not have the circular-ones property for rows and columns, a matrix in the set $\mathcal{F}_{\text{circRC}} \cup \mathcal{F}^t_{\text{circRC}}$ contained in M as a configuration M can be found in linear time.

Circular-ones property for rows and columns

Our characterization states that the minimal forbidden submatrices for the circular-ones property for rows and columns, apart from 10 sporadic matrices and up to permutation of rows and of columns and transpositions, are

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From this, we prove the following algorithmic consequence for concave-round graphs.

Theorem

There exists an algorithm that, given a graph G that is not concave-round, finds in linear time either an induced subgraph of G which is a minimal forbidden induced subgraph for the class of concave-round graphs or a chordless odd cycle in $\overline{G}$.

Concave-round graphs: forbidden subgraphs detection

As a result of all this analysis, we can now conclude the following:

Corollary

Given a graph G that is not concave-round, it is possible to find an induced subgraph of G which is a minimal forbidden induced subgraph for the class of concave-round graphs in linear time.

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Sketch of the proof.

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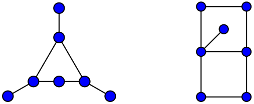
Sketch of the proof.

- By the preceding theorem, let C an odd chordless cycle in $\overline{G}$.
- Since G is not concave-round, G contains a minimal forbidden induced subgraph F for the class of proper circular-arc graphs (Tucker, 1974). We can find F using Soulignac (2015). If $\overline{F}$ is not:



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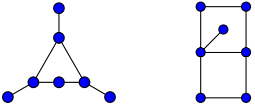

  we are done. Hence, we assume that $\overline{F}$ is one of the above graphs.
- Our solution to the problem by Bang-Jensen et al. (2000) shows how to find a minimal forbidden induced subgraph for the class of concave-round graphs given F and C. □

Connections to other circular-arc graphs

Theorem (Tucker, 1970)

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We observe, by combining different results by Tucker (1971 and 1974), Müller (1997) and Hell and Huang (2004), that the above theorem extends to concave-round graphs.

Corollary

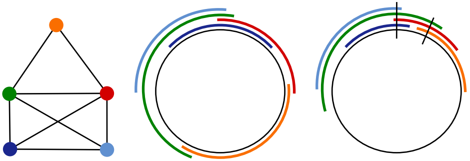
Every concave-round graph is a normal circular-arc graph.

(The result from Hell and Huang (2004) we use is that the complement of an interval bigraph is a normal circular-arc graph.)

Connection to Helly circular-arc graphs

Helly circular-arc graphs (Gavril, 1974)

A graph is a Helly circular-arc graph if it is the intersection graph of a set of arcs on a circle such that all the arcs corresponding to vertices of a clique share a point in the circle.



Minimal circular-arc claw-free obstacles

- Lin and Szwarcfiter (2006) characterized, by forbidden induced subgraphs, called obstacles, those circular-arc graphs that are Helly circular-arc graphs. Obstacles are not necessary minimal (may contain other obstacles) and also are not necessarily circular-arc.

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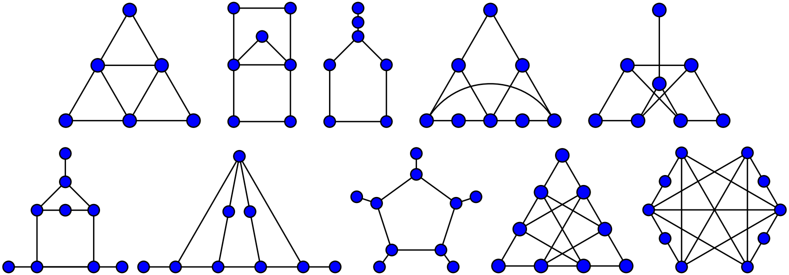
Theorem

The minimal circular-arc claw-free obstacles are $\overline{3K_2}$, $\overline{P_7}$, $\overline{2P_4}$ and the complement of the following graphs:



Connection to Helly circular-arc graphs

Combining results by Lin and Szwarcfiter (2006) with our results, we get:

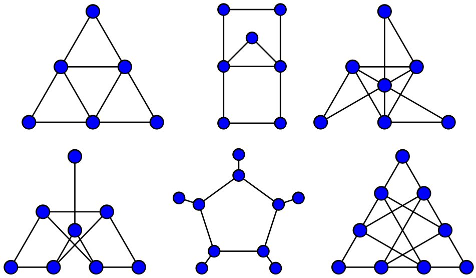
Corollary

For each graph G, the following assertions are equivalent:

(i) G is a Helly circular-arc graph and quasi-line (i.e., $\overline{C_{2k+1} \cup K_1}$-free for every $k \geqslant 1$);

(ii) G is a Helly circular-arc graph and concave-round;

(iii) G contains none of the following as induced subgraphs: $C_k \cup K_1$ for any $k \geqslant 4$, $\overline{C_{2k+1} \cup K_1}$ for any $k \geqslant 1$, $\overline{C_6}$, $\overline{3K_2}$, $\overline{2P_4}$, $\overline{P_7}$, plus:



Thank you very much for your attention!