The strict terminal connection problem with a bounded number of routers

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VII Latin American Workshop on Cliques in Graphs, 2016

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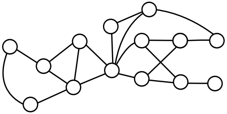

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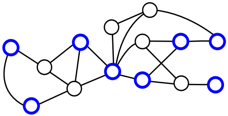

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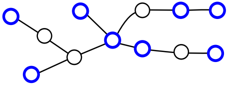

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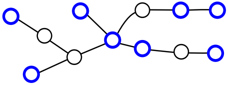

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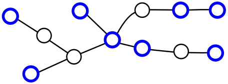

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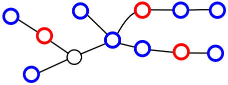

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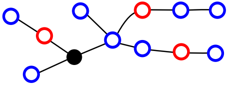

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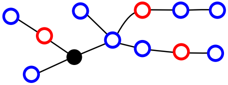

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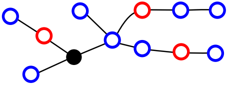

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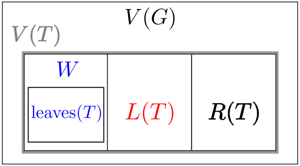

The terminal connection problem

- Motivated by applications in networks, Dourado et al. proposed the TERMINAL CONNECTION PROBLEM:

TERMINAL CONNECTION PROBLEM (TCP)

Instance: A connected graph G, a subset $W \subseteq V(G)$ and two non-negative integers ℓ and r.

Question: Does G admit a connection tree on W with $|L(T)| \leq \ell$ and $|R(T)| \leq r$?

Dourado C. M., Oliveira A. R., Protti, F., and Souza, S. U.: Design of connection networks with bounded number of non-terminal vertices, **Latin-American Workshop on Clique in Graphs 2012**, *Matemática Contemporânea*, 42, pp. 39–48, 2014.

TCP: computatinal complexity

- The TCP is NP-complete even when either ℓ or r is bounded by a constant.

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- **Notation.** ℓ constant: TCP(ℓ); r constant: TCP(r); ℓ and r constants: TCP(ℓ, r).

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STRICT TERMINAL CONNECTION PROBLEM (S-TCP)

Instance: A connected graph G, a subset $W \subseteq V(G)$ and two non-negative integers ℓ and r.

Question: Does G admit a **strict** connection tree T on W with $|L(T)| \leq \ell$ and $|R(T)| \leq r$?

Dourado C. M., Oliveira A. R., Protti, F., and Souza, S. U.: Conexão de terminais com número restrito de roteadores e elos, In Annals of XLVI SBPO, pp. 2965–2976, 2014.

S-TCP: computational complexity

- The S-TCP(ℓ, r) is polynomial-time solvable.
- The S-TCP(ℓ) is NP-complete.

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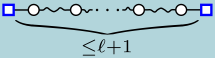
Lemma

If $|W| = 2$, then $\mathcal{I} = (G, W, \ell)$ is a Yes *of the* S-TCP*($r = 0$) if and only if the distance in G between the two vertices of W is at most $\ell + 1$.*



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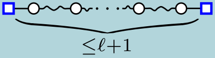

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Let T be a tree. If $|\text{leaves}(T)| > 2$, then $\Delta(T) \geq 3$.

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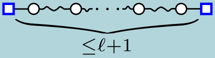

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Corollary

The S-TCP*($r = 0$) is polynomial-time solvable.*

S-TCP($r = 1$)

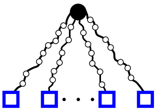

Figure : The topology.

S-TCP($r = 1$): MIN-SUM st−DISJOINT PATHS

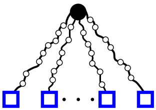

Figure : The topology.

- We prove that the S-TCP($r = 1$) is polynomial-time solvable by a Turing reduction to MIN-SUM st−DISJOINT PATHS:

MIN-SUM st−DISJOINT PATHS (st-DP)

Instance: A graph G, two distinct vertices $s, t \in V(G)$ and two non-negative integers k and x.

Question: Are there k vertex-disjoint paths between s and t in G whose sum of their lenghts is at most x?

S-TCP($r = 1$): the instance $f(\mathcal{I}, \rho)$

- Let $\mathcal{I} = (G, W, \ell)$ be an instance of the S-TCP($r = 1$) and let $\rho \in V(G) \setminus W$.

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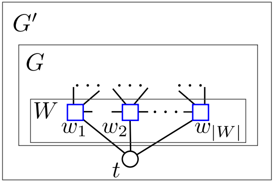

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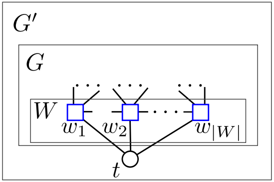
  - $s = \rho$, $k = |W|$ e $x = \ell + 2k$.



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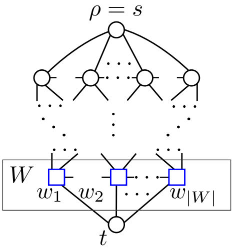

S-TCP($r = 1$): relation between $\mathcal{I}$ and $f(\mathcal{I}, \rho)$

Lemma

An instance $\mathcal{I} = (G, W, \ell)$ of the S-TCP($r = 1$) is a YES *instance if and only if there is $\rho \in V(G) \setminus W$, such that $f(\mathcal{I}, \rho)$ is a* YES *instance of st-DP.*

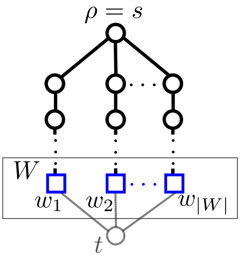

Figure : $k = |W|$ e $x = \ell + 2k$

The S-TCP($r = 1$) is polynomial-time solvable

- For all $\rho \in V(G) \setminus W$:
  - Construct the instance $f(\mathcal{I}, \rho)$;
  - Verify if $f(\mathcal{I}, \rho)$ is a YES instance of $st-$DP;
  - If so: return $\mathcal{I}$ is a YES instance of the S-TCP($r = 1$);
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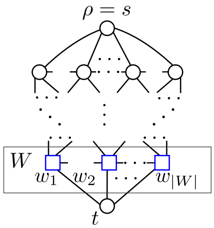

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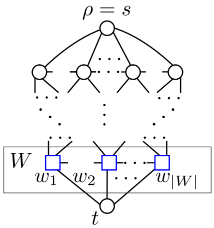

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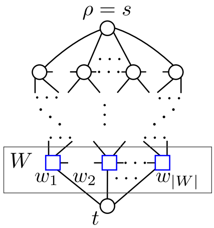

Corollary

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The S-TCP($r = 2$) when the two routers are neighbours is polynomial-time solvable.

The PARTITIONED S-TCP($r = 2$)

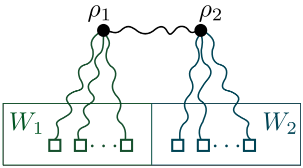
PARTITIONED STRICT TERMINAL CONNECTION PROBLEM

Instance: A connected graph G, a subset $W \subseteq V(G)$, a bipartition $W = W_1 \cup W_2$ and a non-negative integer ℓ.

Question: Does G admit a strict connection tree T on W with $|L(T)| \leq \ell$ and $|R(T)| = 2$, such that:

- $\text{dist}_T(\rho_1, w) < \text{dist}_T(\rho_2, w), \forall\, w \in W_1$;
- $\text{dist}_T(\rho_2, w') < \text{dist}_T(\rho_1, w'), \forall\, w' \in W_2$;

where $R(T) = \{\rho_1, \rho_2\}$?

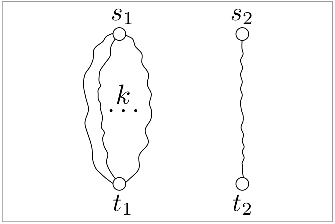


$k+1$DP

$k+1$ VERTEX-DISJOINT PATHS BETWEEN TWO PAIRS OF VERTICES ($k+1$DP)

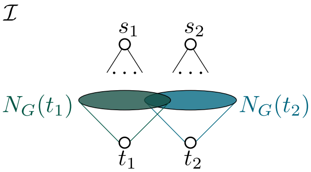
Instance: A graph G, four vertices $s_1, t_1, s_2, t_2 \in V(G)$ and an integer $k \geq 0$.

Question: Are there $k+1$ vertex-disjoint paths in G, such that k of these paths are between s_1 and t_1 and the remaining path is between s_2 and t_2?



PARTITIONED S-TCP($r = 2$): the instance $f(\mathcal{I})$

- Let $\mathcal{I} = (G, (s_1, t_1), (s_2, t_2), k)$ be an instance of $k + 1$DP;



Partitioned S-TCP($r = 2$): the instance $f(\mathcal{I})$

- Let $\mathcal{I} = (G, (s_1, t_1), (s_2, t_2), k)$ be an instance of $k + 1$DP;
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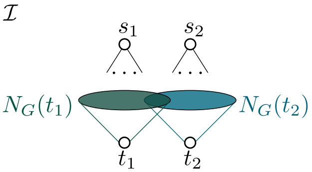

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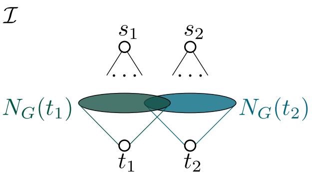

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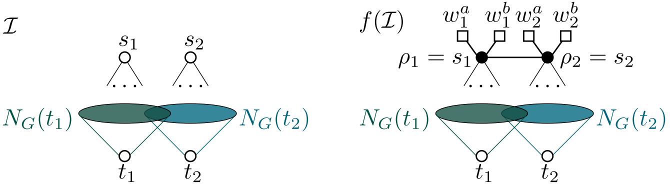

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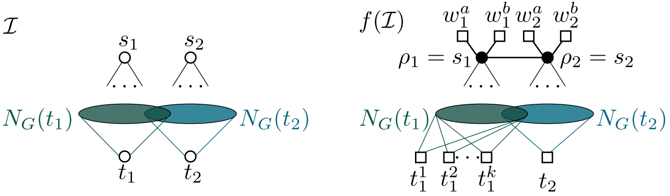

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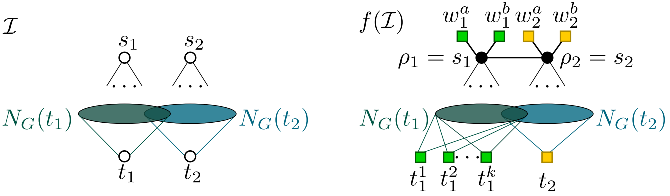
- Let $f(\mathcal{I}) = (G', W_1 \cup W_2, \ell)$ be the instance of the PARTITIONED S-TCP($r = 2$) defined as follows:



- $W_1 = \{t_1^i \mid 1 \leq i \leq k\} \cup \{w_1^a, w_1^b\}$ e $W_2 = \{w_2^a, w_2^b, t_2\}$; $\ell = |V(G) \setminus W|$.

The PARTITIONED S-TCP($r = 2$) is NP-complete

Lemma

An instance $\mathcal{I}$ of $k+1$DP is a YES *instance if and only if $f(\mathcal{I})$ is a* YES *instance of the* PARTITIONED S-TCP*($r = 2$).*

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Lemma

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Corollary

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Corollary

*For all $r \geq 2$, the r-*PARTITIONED S-TCP*(r) is NP-complete.*

Conclusion

Results:

- The S-TCP($r = 0$) , S-TCP($r = 1$) and the S-TCP($r = 2$) when is required that the two router are neighbours are polynomial-time solvable;
- The r-PARTITIONED S-TCP(r) is NP-complete, for all $r \geq 2$.

Open problem:

- The computational complexity of the S-TCP(r), for $r \geq 2$.

Thank you for your attention!

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