Locating-Dominating Partitions in Graphs [1]

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[1]Joint work with **Carmen Hernando** and **Mercè Mora**.

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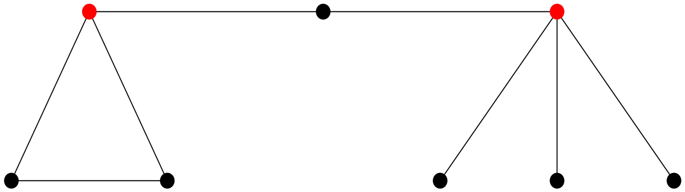

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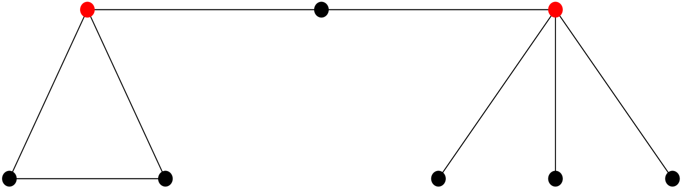

- $\boxed{\gamma(G) = 2}$ (red vertices form a γ-code).

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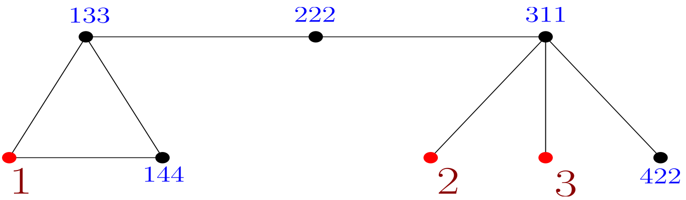

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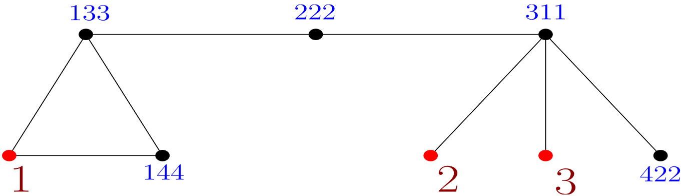

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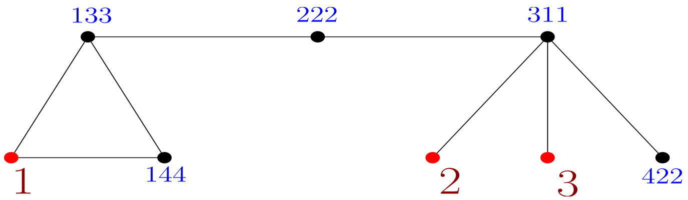

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- Note that vertices 222 and 422 are not dominated by S.

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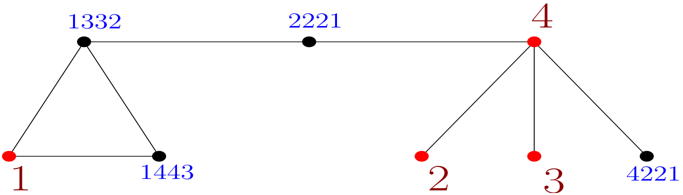

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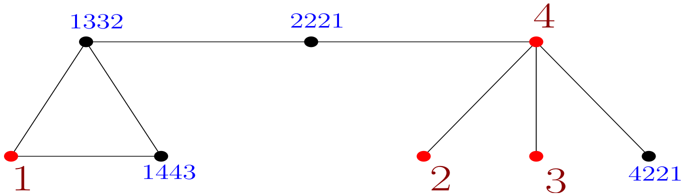

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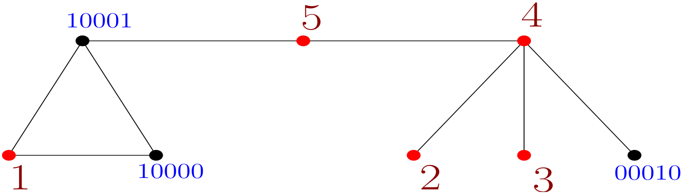

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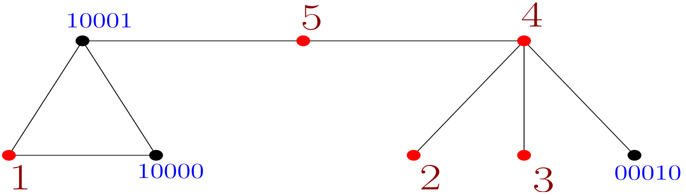

- $\boxed{\lambda(G) = 5}$ ($S = \{1, 2, 3, 4, 5\}$ is a λ-code).

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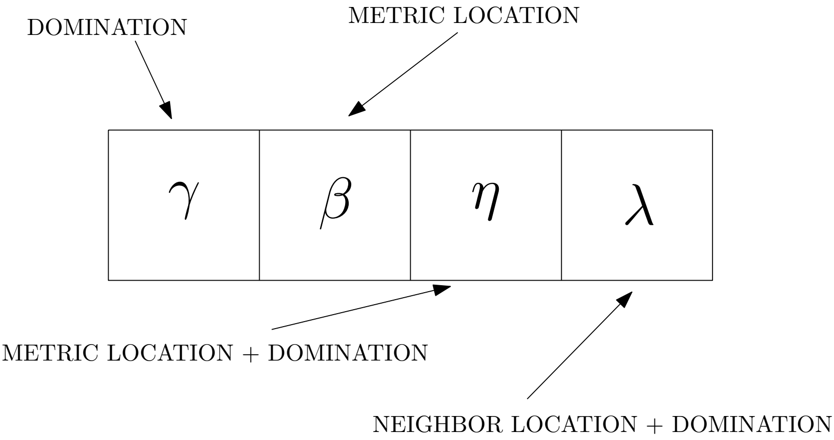

$$\max\{\gamma, \beta\} \leq \eta \leq \min\{\gamma + \beta, \lambda\}$$

▷ A **partition** $\Pi = \{S_1, \dots, S_k\}$ of V *dominates* G if, for every $i \in \{1, \dots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \dots, k\}$,

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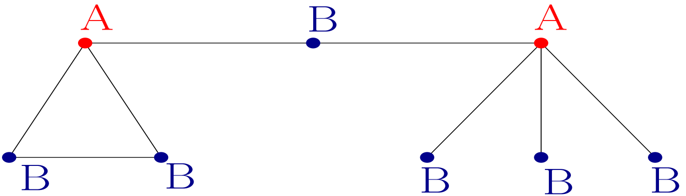

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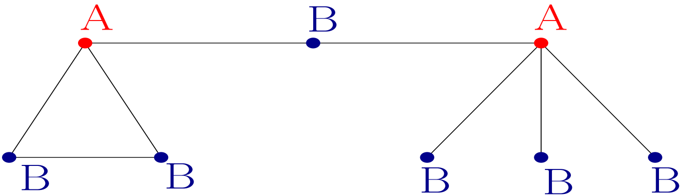

- $\gamma_p(G) = 2$ (TRUE for every graph).

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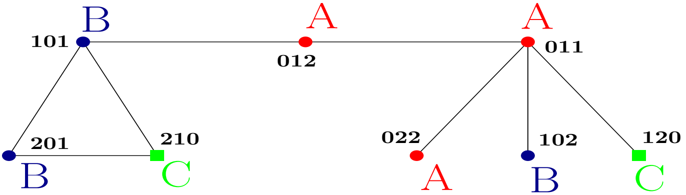

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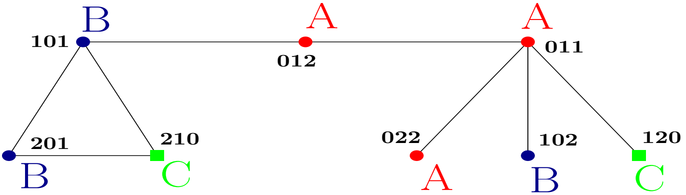

- $\beta_p(G) = \tau(G) = 3$ ($\Pi = \{A, B, C\}$ is a β_p-partition).

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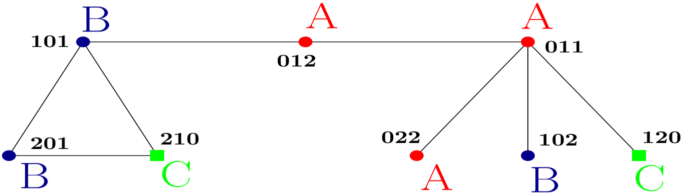

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→ Π is not dominating (**022** is an internal vertex of part A).

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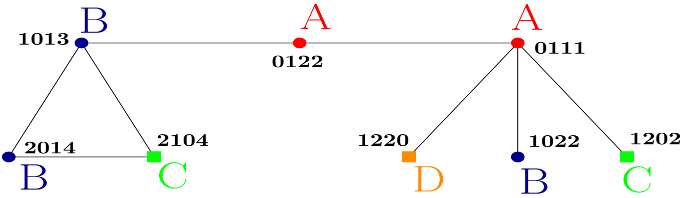

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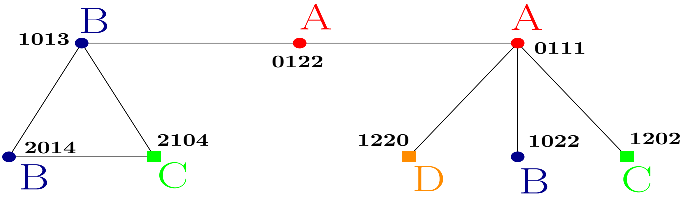

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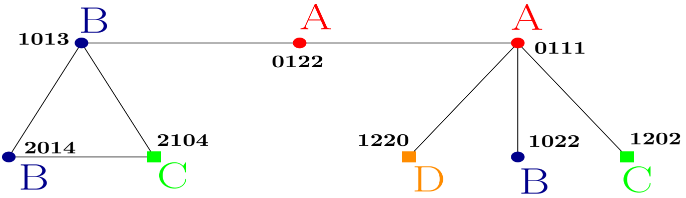

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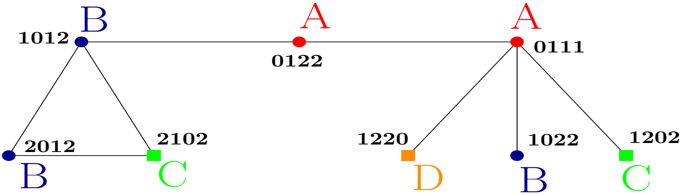

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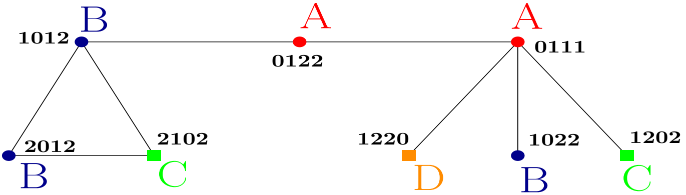

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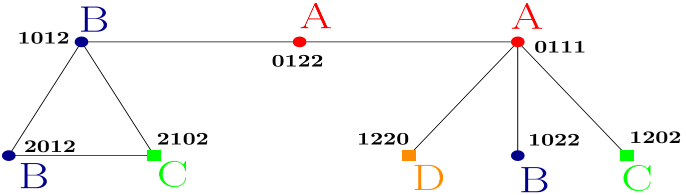

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- $\beta_p(P_{10}) = 2$, $\eta_p(P_{10}) = 3$, $\lambda_p(P_{10}) = 4$

$\gamma \leq \eta$

$\gamma_p = 2$

$$\begin{array}{ccccc}
 & & \beta_p + 1 & & \\
 & & \vee\!\mid & & \\
\beta_p & \leq & \eta_p & \leq & \lambda_p \\
\mid\!\wedge & & \mid\!\wedge & & \mid\!\wedge \\
\beta + 1 & \leq & \eta + 1 & \leq & \lambda + 1 \\
 & & \mid\!\wedge & & \\
 & & \gamma + \beta + 1 & &
\end{array}$$

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▷ $D = C_1 \cup \ldots \cup C_k$.

▷ $\Pi' = \{S_1, \ldots, S_k, D\}$ MLD-partition.

- $\beta_p = n$: K_n.

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- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

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- $\eta_p = n-1 \Leftrightarrow \lambda_p = n-1$: $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.

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▷ **Discrete Math., 308 (2008), 5026–5031.** (wrong result: 22 families)

- $\beta_p = n$: K_n.
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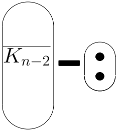
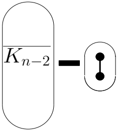
▷ **Discrete Math., 308 (2008), 5026–5031.** (wrong result: 22 families)

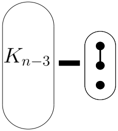
- $\eta_p = n-2 \Leftrightarrow \lambda_p = n-2$: $\{H_i\}_{i=1}^{\mathbf{17}}$.



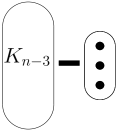
$H_1 \cong K_{2,n-2}$



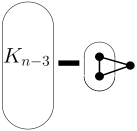
$H_2 \cong \overline{K_{n-2}} \vee K_2$



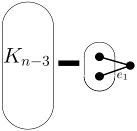
$H_3 \cong K_{n-3} \vee (K_2 + K_1)$



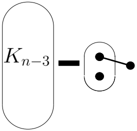
$H_4 \cong K_{n-3} \vee \overline{K_3}$



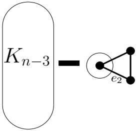
$H_5 \cong (K_{n-3} + K_1) \vee K_2$



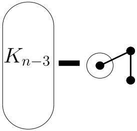
$H_6 \cong (K_{n-3} + K_1) \vee \overline{K_2}$



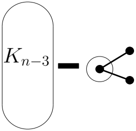
$H_7 \cong H_6 - e_1$



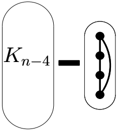
$H_8 \cong (K_{n-3} + K_2) \vee K_1$



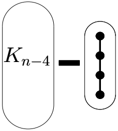
$H_9 \cong H_8 - e_2$



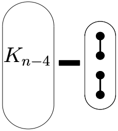
$H_{10} \cong (K_{n-3} + \overline{K_2}) \vee K_1$



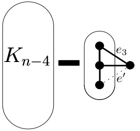
$H_{11} \cong K_{n-4} \vee C_4$



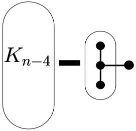
$H_{12} \cong K_{n-4} \vee P_4$



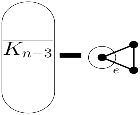
$H_{13} \cong K_{n-4} \vee 2K_2$



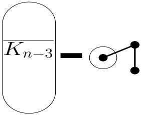
$H_{14} \cong (K_{n-4} + K_1) \vee P_3 - e'$



$H_{15} \cong H_{14} - e_3$



$H_{16} \cong K_1 \vee (\overline{K_{n-3}} + K_2)$



$H_{17} \cong H_{16} - e$

[5] also called *stable partition* or *proper coloring*.

▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is a *coloring partition* [5] of G, if for every $i \in \{1, \ldots, k\}$, S_i is an independent set.

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$$\begin{array}{|ccccl|}
\hline
\gamma_p & \longmapsto & \gamma_p^i & = & \chi \quad \text{chromatic number} \\
 & & & & \\
\beta_p & \longmapsto & \beta_p^i & & \\
\geq & & \| & & \\
\eta_p & \longmapsto & \eta_p^i & = & \chi_{ML} \quad \text{ML-chromatic number} \\
\geq & & \geq & & \\
\lambda_p & \longmapsto & \lambda_p^i & = & \boxed{\chi_{NL}} \quad \text{NL-chromatic number} \\
\hline
\end{array}$$

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$$\begin{array}{lcccl}
\gamma_p & \longmapsto & \gamma_p^i & = \quad \chi & \text{chromatic number} \\
 & & & & \\
\beta_p & \longmapsto & \beta_p^i & & \\
\mid\wedge & & \| & & \\
\eta_p & \longmapsto & \eta_p^i & = \quad \chi_{ML} & \text{ML-chromatic number} \\
\mid\wedge & & \mid\wedge & & \\
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\end{array}$$

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γ_p	$\longmapsto$	γ_p^i	=	χ	chromatic number
β_p	$\longmapsto$	β_p^i			
$\lvert\wedge$		$\lvert\lvert$			
η_p	$\longmapsto$	η_p^i	=	χ_{ML}	ML-chromatic number
$\lvert\wedge$		$\lvert\wedge$			
λ_p	$\longmapsto$	λ_p^i	=	$\boxed{\chi_{NL}}$	NL-chromatic number

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- $\boxed{n \leq k \cdot [d^{k-1} - (d-1)^{k-1}]}$, where $k \in \{\eta_p, \chi_{ML}\}$.

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γ_p	$\longmapsto$	γ_p^i	$=$	χ	chromatic number
β_p	$\longmapsto$	β_p^i			
$\mid\wedge$		$\Vert$			
η_p	$\longmapsto$	η_p^i	$=$	χ_{ML}	ML-chromatic number
$\mid\wedge$		$\mid\wedge$			
λ_p	$\longmapsto$	λ_p^i	$=$	$\boxed{\chi_{NL}}$	NL-chromatic number

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- $\boxed{n \leq k \cdot [2^{k-1} - 1]}$, where $k \in \{\lambda_p, \chi_{NL}\}$.

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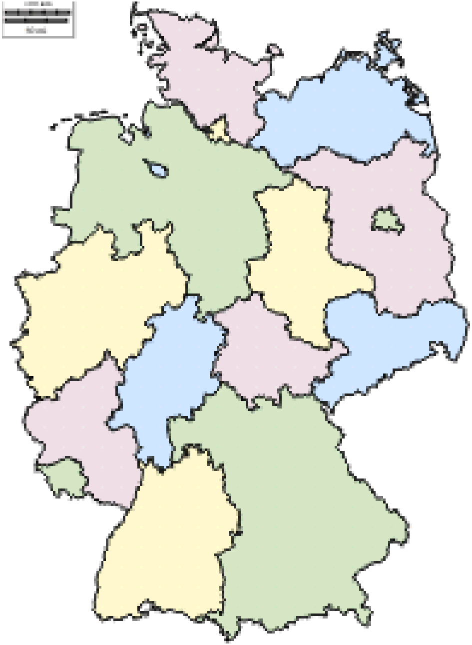
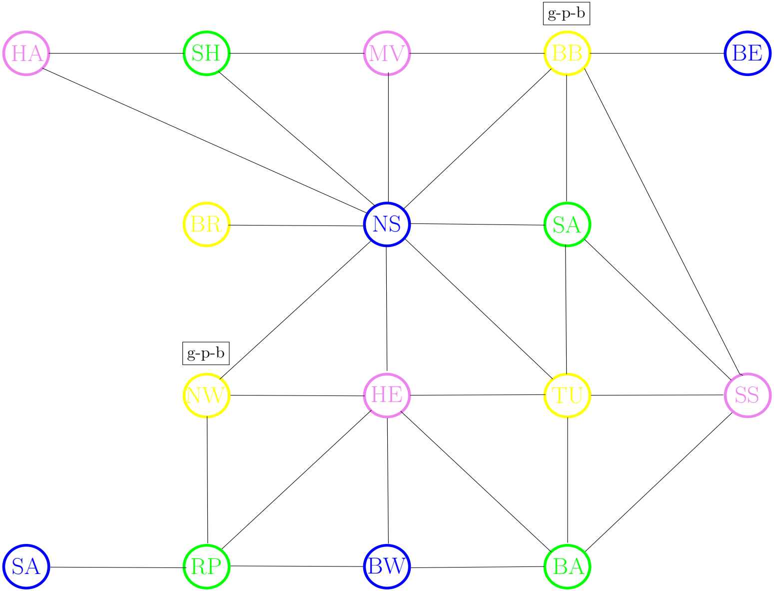
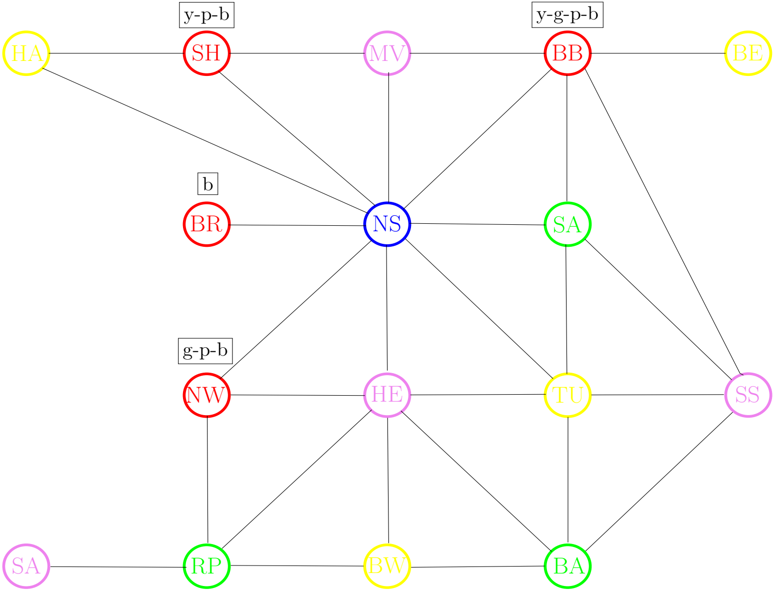
$n = 13 + 3 = 16$
$m = 29$
$D = 5$
$\delta = 1$
$\Delta = 9$



$\chi = 4$



$$\chi_{NL} = 5$$



Google, Yahoo: arxiv pelayo locating partition

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