On Clique Corona Graphs - A Short Survey

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November 8-11, 2016
VII Latin American Workshop on Cliques in Graphs
La Plata, Argentina

Outline

- Cliques and independent sets
- Corona of graphs and clique corona graphs
- Well-covered graphs and clique corona graphs
- Independence polynomials of clique corona graphs
- Greedoids on clique corona graphs
- Some conclusions ...

Cliques and complete graphs

Definition

- A clique in G is a set of pairwise **adjacent vertices**.

The clique number the size $\omega(G)$ of a largest (i.e., **maximum**) clique in G.

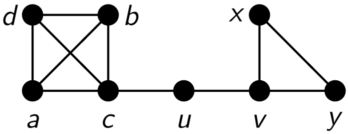
- If all n vertices of $G = (V, E)$ are pairwise adjacent, then $G = K_n$, i.e., G is a **complete graph**.

Example

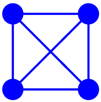
$\{u\}, \{a, b\}, \{u, v\}, \{x, y, v\}, \{a, b, c, y\}$ are cliques of G

$\{a, b, c, d\}$ is a maximum clique, hence $\omega(G) = 4$

G



K_4



Independent sets and independence number

Definition

An independent set is a set of pairwise **non-adjacent vertices**.

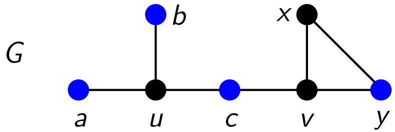
The independence number of G is the size $\alpha(G)$ of a largest (i.e., **maximum**) independent set in G.

Example

$\{a\}, \{a,b\}, \{a,b,x\}, \{a,b,c,y\}$ are independent sets of G

$\{u,v\}, \{u,x\}, \{u,y\}, \{a,b,c,y\}$ are maximal independent sets of G

$\{a,b,c,x\}, \{a,b,c,y\}$ are maximum independent sets, hence $\alpha(G) = 4$



Corona of two graphs

Definition (R. Frucht, F. Harary, Aequationes Math. 1970)

The **corona** $G \circ H$ of G and H is the disjoint union of G and $|V(G)|$ copies H, with additional edges joining each vertex $v \in V(G)$ to all the vertices of a copy of H.

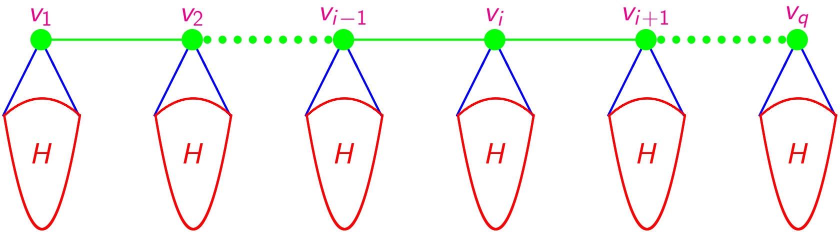


Figure: The corona $P_q \circ H$.

Corona of a graph and a family of graphs

Definition

The **corona** $G \circ \mathcal{H}$ of G and the family $\mathcal{H} = \{H_v : v \in V(G)\}$ is the disjoint union of G and $H_v, v \in V(G)$, with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v.

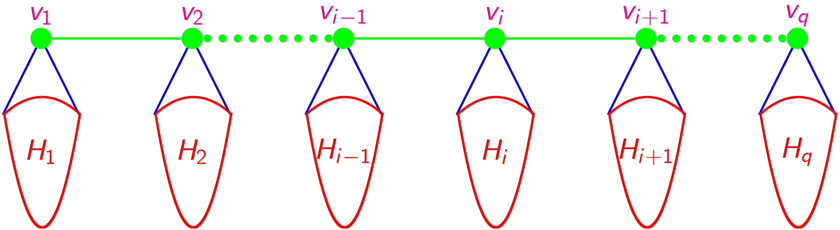


Figure: The corona $P_q \circ \{H_1, H_2, ..., H_q\}$.

Clique Corona Graphs

Definition

The **corona** $G \circ \mathcal{H}$ of G and a family $\mathcal{H} = \{H_v : v \in V(G)\}$ of pairwise disjoint complete graphs $K_{n_v}, v \in V(G)$, is called a **clique corona graph**.

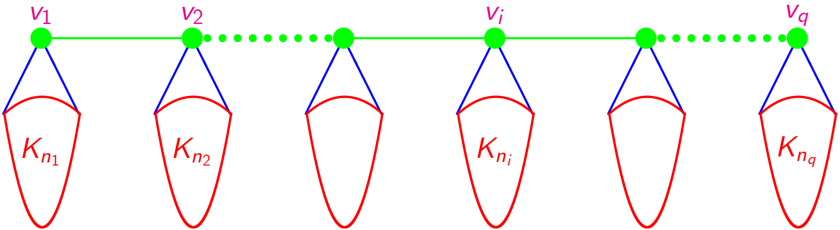


Figure: The corona $P_q \circ \left\{K_{n_1}, K_{n_2}, ..., K_{n_q}\right\}$.

Clique Corona Graph

Definition

The corona of G and a complete graph, say K_n,

is a **clique corona graph** denoted $G \circ K_n$.

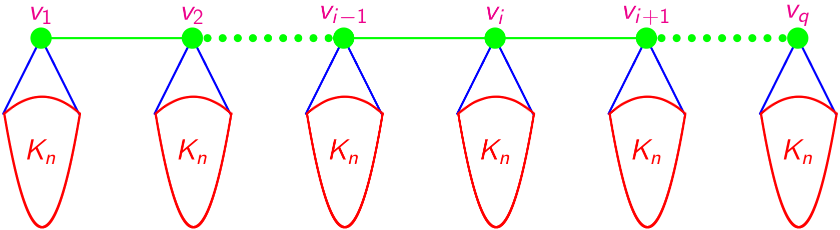


Figure: The **clique corona** $P_q \circ K_n$.

Well-Covered Graphs

&

Clique Corona Graphs

Well-Covered Graphs

Definition (M. D. Plummer, J Comb Th 1970)

G is **well-covered** if all its maximal independent sets have the same size.

- The only well-covered cycles are C_3, C_4, C_5 and C_7.
- **girth** $=$ the length of a **shortest cycle,** and $= \infty$ if **no** cycles.

Theorem (A. Finbow, B. Hartnell, R. Nowakowski, J Comb Th 1993)

If G is a connected graph of girth ≥ 6, and $C_7 \neq G \neq K_1$, then

G *is* **well-covered if and only if** $G = H \circ K_1$ *for some graph* H,

i.e., G *is under the form of a* **clique corona graph**.

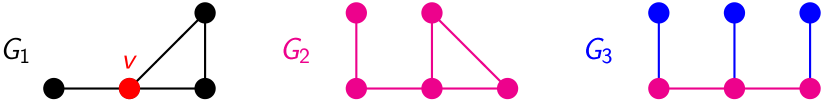


Figure: Only G_1 is **NOT** well-covered. $G_3 = P_3 \circ K_1$

Very Well-Covered Graphs

Definition (O. Favaron, Discrete Mathematics 1982)

If G is well-covered, has no isolated vertices, and $|V(G)| = 2\alpha(G)$, then G is a **very well-covered** *graph*.

- The only very well-covered cycle is C_4.

Theorem (Levit and Mandrescu, Congressus Numerantium 2007)

A connected graph G of girth ≥ 5 is **very well-covered iff** $G = H \circ K_1$ *for some graph H,* **i.e.,** *G is under the form of a* **clique corona graph**.

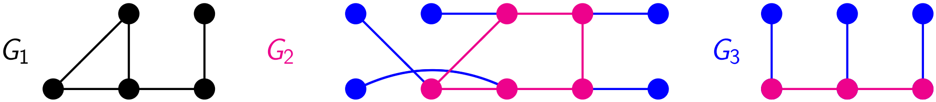


Figure: G_1 is well-cov but **not** very well-cov, $G_2 = C_5 \circ K_1$ and $G_3 = P_3 \circ K_1$

1-Well-Covered Graphs

Definition (J. W. Staples, Ph.D. Thesis, 1975)

A **well-covered** graph G (with at least two vertices) is 1-**well-covered** if $G - v$ is well-covered for every vertex v.

- The only 1-well-covered cycles are C_3 and C_5.
- P_4 is very well-covered, but **NOT** 1-well-covered.
- K_2 is **both** very well-covered and 1-well-covered.

Example

G_1 is well-covered; G_2 is very well-covered; **only** G_3 is 1-well-covered.

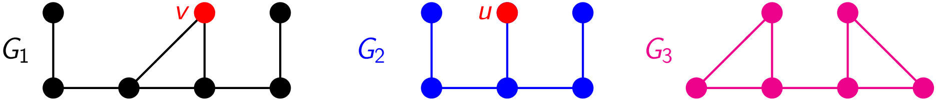


Figure: $G_1 - v$ is **not** well-covered, $G_2 - u$ is **not** well-covered

Theorem (J. Topp, L. Volkman, Ars Combinatoria 1992)

The corona $G \circ \mathcal{H}$ *of* G *and* $\mathcal{H} = \{H_v : v \in V(G)\}$ *is* **well-covered** *if and only if each* $H_v \in \mathcal{H}$ *is a* **complete graph** *on at least* **one** *vertex.*

Theorem (Levit and Mandrescu, 2016 - For G without isolated vertices.)

The corona $G \circ \mathcal{H}$ *of* G *and* $\mathcal{H} = \{H_v : v \in V(G)\}$ *is* 1-**well-covered** *if and only if each* $H_v \in \mathcal{H}$ *is a* **complete graph** *on at least* **two** *vertices.*

Example

$G_1 = P_2 \circ \{K_1, 2K_1\}$, $G_2 = P_2 \circ \{K_1, K_2\}$, $G_3 = P_2 \circ \{K_2, K_3\}$

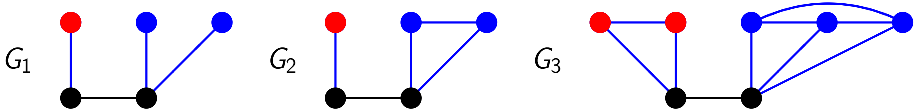


Figure: G_1 is **not** well-covered, G_2 is well-covered, while G_3 is 1-**well-covered**.

Independence Polynomials of Clique Corona Graphs

Some facts about polynomials

- Let $P(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial with all $a_k > 0$

P has all its roots real

I. Newton, Arithmetica universalis, 1707

Newton's inequality: $a_k^2 \geq a_{k-1} \bullet a_{k+1} \bullet \frac{k+1}{k} \bullet \frac{n-k+1}{n-k}$ for $1 \leq k \leq n-1$

P is log-concave, i.e., $a_k^2 \geq a_{k-1} \bullet a_{k+1}$ for $1 \leq k \leq n-1$

folklore

P is unimodal, i.e., $a_0 \leq \ldots \leq a_j \geq \ldots \geq a_n$ for some j (= mode)

Independence polynomial of a graph

- If G has s_k independent sets of size k, then

$$I(G;x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

  is the independence polynomial of G (I. Gutman & F. Harary, 1983).

- **Examples:**

Figure: $I(G;x) = 1 + 4x + 3x^2 + x^3$, while $I(H;x) = 1 + 4x + 2x^2$.

- $I(G \cup H; x) = (1 + 4x + 3x^2 + x^3) \bullet (1 + 4x + 2x^2)$
- $I(G + H; x) = (1 + 4x + 3x^2 + x^3) + (1 + 4x + 2x^2) - 1$, where $G + H =$ the *Zykov sum* of G and H, i.e., the graph obtained by joining **each vertex** of G with **each vertex** of H by an edge.

Definition

Given a graph G, its **line graph** $L(G)$ is the graph whose vertex set is the edge set of G, and two vertices are adjacent if they share an end in G.

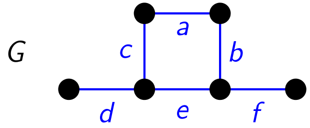
- **Matching polynomial** $M(G;x)$ of $G \equiv$ **Independence poly** of $L(G)$
- Unlike $M(G;x)$, the polynomial $I(G;x)$ may have **non-real roots**.
- $L(G)$ is a **claw-free** graph (i.e., a $K_{1,3}$-free graph)!

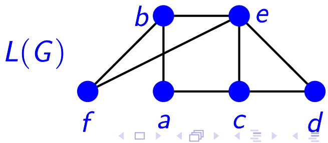
Theorem (M. Chudnovsky and P. Seymour, J. Comb Th 2007)

If G is a claw-free graph, then $I(G;x)$ has only real roots.

Example

$$M(G;x) = I(L(G);x) = 1 + 6x + 7x^2 + x^3.$$





Corona of graphs and its independence polynomial

Theorem (I. Gutman, Publications de l'Institute Mathematique, 1992)

$I(G \circ H; x) = (I(H; x))^n \bullet I\left(G; \frac{x}{I(H;x)}\right)$, *with* $n = |V(G)|$.

Figure: $G = P_3, H = K_2$ and $G \circ H = P_3 \circ K_2$.

Example: $I(P_3 \circ K_2; x) = (I(K_2; x))^3 \cdot I\left(P_3; \frac{x}{I(K_2;x)}\right) =$

$$= (1 + 2x)^3 \cdot I\left(P_3; \frac{x}{1+2x}\right) = 1 + 9x + 25x^2 + 22x^3$$

- $V(P_n) = \{v_i : 1 \leq i \leq n\}, \quad E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$, and all the vertices of K_m are joined to v_i, where $i \in \{1, 2, ..., n\}$.

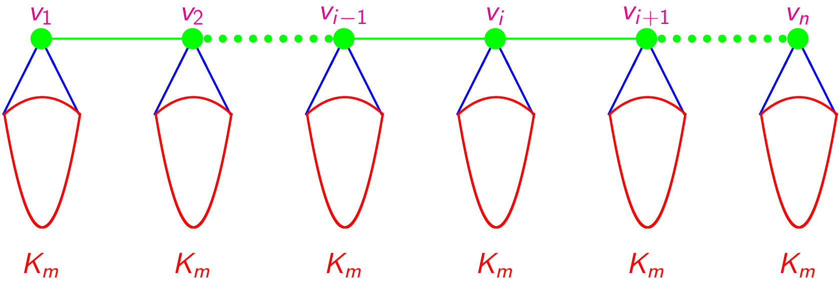


Figure: The **clique corona** $G = P_n \circ K_m$.

- $$I(P_n \circ K_m; x) = (I(K_m; x))^n \bullet I\left(P_n; \frac{x}{I(K_m;x)}\right) =$$
$$= (1 + mx)^n \bullet I\left(P_n; \frac{x}{1+mx}\right)$$

Basic questions about independence polynomials

Which necessary / sufficient conditions on G guarantee that $I(G; x)$

- has only real roots ?
- is log-concave ? or is unimodal ?

Theorem (Y. Alavi, P. J. Malde, A. J. Schwenk and P. Erdös, 1987)

For every permutation π of $\{1, 2, ..., \alpha\}$, there is a graph G with $\alpha(G) = \alpha$, such that $s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)}$.

Conjecture (Y. Alavi, P. J. Malde, A. J. Schwenk and P. Erdös, 1987)

$I(T; x)$ is unimodal for every tree T.

Which graph operations lead to graphs whose independence polynomials

- have only real roots ?
- are log-concave ? or are unimodal ?

Conjecture (J. I. Brown, K. Dilcher, R. J. Nowakowski, Journal of Algebraic Combinatorics 2000)

The independence polynomial of every well-covered graph is unimodal.

- **TRUE** for $\alpha(G) \leq 3$ and **FALSE** for $4 \leq \alpha(G) \leq 7$

  (T. S. Michael and W. N. Traves, Graphs & Combinatorics 2003)
- **FALSE** for $\alpha(G) \geq 4$ (Levit & Mandrescu, European J Comb 2006)

Conjecture (Levit and Mandrescu, European J Comb 2006)

The indep polynomial of every **very well-covered** *graph is unimodal.*

- **TRUE** for very well-covered graphs with $\alpha(G) \leq 9$

  (Levit and Mandrescu, in *Graph Theory in Paris 2006*)
- $I(G;x)$ is **log-concave** for very well-covered graphs with $\alpha(G) \leq 5$

  (Levit and Mandrescu, in *Graph Theory in Paris 2006*)

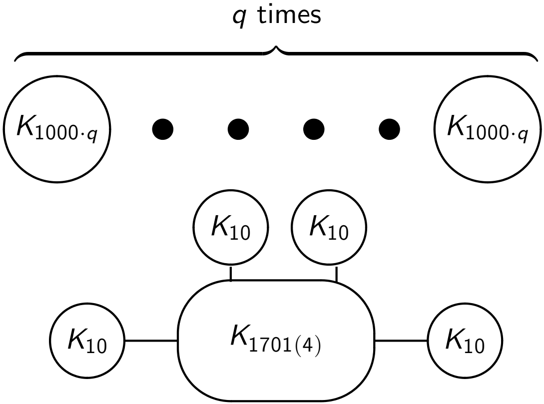
Fact (T. S. Michael, W. N. Traves, Graphs & Comb 2003)

There are well-cov graphs with non-unimodal independence polynomials.

Examples (Levit and Mandrescu, European J. Comb 2006)

$G_q = (qK_{1000 \cdot q}) \cup ((4K_{10}) + K_{1701(4)})$ and $G_q + G_q$ are well-covered graphs, and their independence polynomials are not unimodal.

G_q



Theorem (T. S. Michael and W. N. Traves, Graphs & Comb 2003)

If G is well-covered of order ≥ 2, then $s_0 \leq s_1 \leq \cdots \leq s_{\lceil \alpha(G)/2 \rceil}$.

Conjecture (Roller-Coaster Conjecture - T. S. Michael, W. N. Traves, Graphs & Comb 2003)

For any permutation π of $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, ..., \alpha\}$, there exists a well-covered graph G having with $\alpha(G) = \alpha$ such that

$$s_{\pi(\lceil \alpha/2 \rceil)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < \cdots < s_{\pi(\alpha)}.$$

- **TRUE** for $\alpha(G) \leq 7$ (Michael and Traves, Graphs & Comb 2003)
- **TRUE** for $\alpha(G) \leq 11$ (Matchett, The Electronic J Comb 2004)
- **TRUE** for **any** $\alpha(G)$ (Cutler and Pebody, J. Comb. Th. **A 145 - 2017**)

Conjecture (Roller-Coaster Conjecture - T. S. Michael, W. N. Traves, Graphs & Comb 2003)

For any permutation π *of* $\{\lceil\alpha/2\rceil, \lceil\alpha/2\rceil+1, ..., \alpha\}$, *there exists a well-covered graph* G *having with* $\alpha(G)=\alpha$ *such that*

$$s_{\pi(\lceil\alpha/2\rceil)} < s_{\pi(\lceil\alpha/2\rceil+1)} < \cdots < s_{\pi(\alpha)}.$$

- **TRUE** for **any** $\alpha(G)$ (Cutler and Pebody, J Comb Th A 145 - 2017)

Theorem (Levit and Mandrescu, in *Graph Theory in Paris 2006*)

If G *is a* **very well-covered** *graph of order* ≥ 2 *with* $\alpha(G)=\alpha$, *then* $s_0 \leq s_1 \leq \cdots \leq s_{\lceil\alpha/2\rceil}$ *and* $s_{\lceil(2\alpha-1)/3\rceil} \geq \cdots \geq s_{\alpha-1} \geq s_\alpha$.

- In other words, for **very well-covered** graphs, the domain of the **"Roller-Coaster Conjecture"** can be **shortened** to $\{\lceil\alpha/2\rceil, \lceil\alpha/2\rceil+\mathbf{1}, ..., \lceil(2\alpha-1)/3\rceil\}$.

- K_2 is 1-well-covered and $I(K_2;x) = 1 + 2x$ is unimodal.

Theorem (Levit and Mandrescu, 2016)

If G is connected and 1-well-covered, $\alpha = \alpha(G)$, and (s_k) are the coefficients of $I(G;x)$, then the following are true:

$$\textbf{(i)}\quad \frac{s_0}{\binom{\alpha}{0}} \leq \frac{s_1}{2 \cdot \binom{\alpha}{1}} \leq \frac{s_2}{2^2 \cdot \binom{\alpha}{2}} \leq \cdots \leq \frac{s_\alpha}{2^\alpha \cdot \binom{\alpha}{\alpha}}; \qquad (*)$$

$$\textbf{(ii)}\quad s_0 \leq s_1 \leq \cdots \leq s_t \quad \text{where } t = \left\lfloor \frac{2\alpha - 1}{3} \right\rfloor.$$

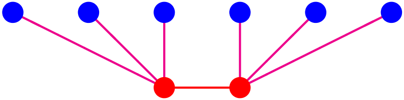
- In other words, for 1-well-covered graphs, the domain of the Roller-Coaster Conjecture, namely $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + \mathbf{1}, ..., \boldsymbol{\alpha}\}$, can be **shortened** to $\{\lceil (2\alpha - 1)/3 \rceil, \lceil (2\alpha - 1)/3 \rceil + \mathbf{1}, ..., \boldsymbol{\alpha}\}$.

Theorem (Levit and Mandrescu, DAM 2016)

Let G be with $E(G) \neq \emptyset$. **If** *$I(G \circ H; x)$ has* **only real roots**,
then *the same is true for* **both** *$I(G; x)$* **and** *$I(H; x)$.*

- The converse is **not** necessarily true; e.g.,
  $I(K_2; x) = 1 + 2x$ and $I(3K_1; x) = (1 + x)^3$ have only real roots,
  while $I(K_2 \circ 3K_1; x) = 1 + 8x + 21x^2 + 26x^3 + 17x^4 + 6x^5 + x^6$
  has **non-real** roots. Notice that: $H = 3K_1$ has $\alpha(H) = 3$.

K_2 $3K_1$



$K_2 \circ 3K_1$

Theorem (Levit & Mandrescu, DAM 2016)

Let G and H be graphs with $E(G) \neq \emptyset$ and $\alpha(H) \leq 2$.
Then $I(G \circ H; x)$ has only real roots **iff** *$I(G; x)$ has only real roots.*

Theorem (E. Mandrescu, Graphs & Combin 2009)

Let G be a connected graph on $n \geq 2$ vertices. Then the following hold:

(i) *$-1/p$ is a root of $I(G \circ K_p; x)$ with the multiplicity $n - \alpha(G) \geq 1$;*

(ii) *there is a bijection between the set of roots of $I(G \circ K_p; x)$ different from $-1/p$ and the set of roots of $I(G; x)$, respecting the multiplicities of the roots; moreover, real roots correspond to real roots.*

- A graph is called *claw-free* if it has no subgraph isomorphic to $K_{1,3}$.

Theorem (M. Chudnovsky and P. Seymour, J Combin Th 2007)

The independence polynomial of a claw-free graph has only real roots.

Corollary

Let G be a connected graph on $n \geq 2$ vertices, and $p \geq 2$. Then

(i) *$G \circ K_p$ is 1-well-covered;*

(ii) *if $I(G; x)$ has* **only real roots** *(e.g., G is claw-free), then $I(G \circ K_p; x)$ has* **only real roots**, *and hence it is* **unimodal**.

- $I(P_n;x) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-j}{j} \cdot x^j$ (J. L. Arocha, 1984).
- $I(P_n;x)$ has **only real roots**, since P_n is claw-free.

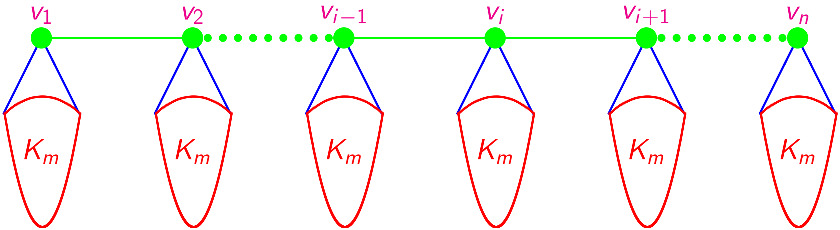


Figure: The **clique corona** $P_n \circ K_m$.

- $I(P_n \circ K_m;x) = (I(K_m;x))^n \bullet I\left(P_n; \frac{x}{I(K_m;x)}\right) =$

$= (1+mx)^n \bullet I\left(P_n; \frac{x}{1+mx}\right)$ has **only real roots**, as $I(P_n;x)$ has.

Local Maximum Stable Sets Greedoids on Vertex Sets of Clique Corona Graphs

Local maximum stable sets

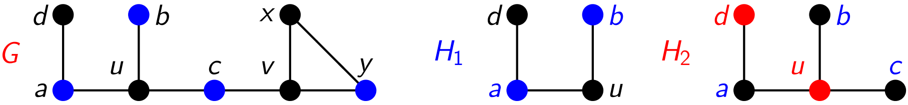
Definition

- $A \subseteq V$ *is a local maximum stable set of* $G = (V, E)$ *if* A *is a maximum stable set in the subgr. induced by its closed neighborhood.*
- $\Psi(G)$ *denotes the family of all local maximum stable sets of* G.

- Each $S \in \Omega(G)$ belongs to $\Psi(G)$ as well. **E.g.**, $\{a, b, c, y\} \in \Psi(G)$.

Example

- $\{a, b\} \in \Psi(G)$, since $\{a, b\}$ is a **max st set** in $H_1 = N[\{a, b\}]$
- $\{d, u\} \notin \Psi(G)$, since $\{a, b, c\}$ is a **max st set** in $H_2 = N[\{d, u\}]$



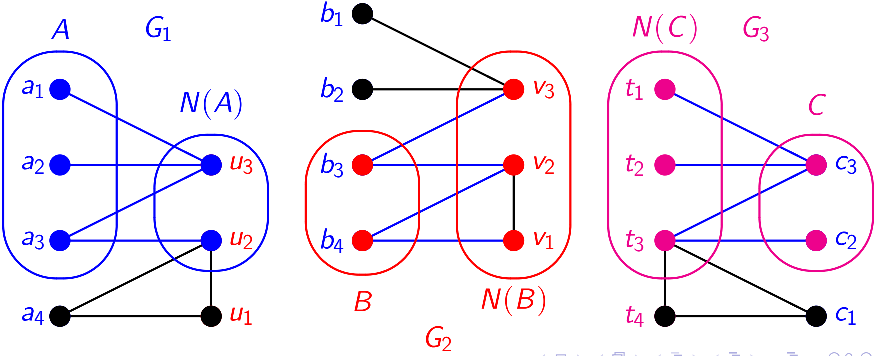
Local maximum stable sets - some more examples

Example

$A = \{a_1, a_2, a_3\} \in \Psi(G_1)$, i.e., A is a local maximum stable set of G_1

$B = \{b_3, b_4\} \in \Psi(G_2)$, i.e., B is a local maximum stable set of G_2

$C = \{c_2, c_3\} \notin \Psi(G_3)$, i.e., C is NOT a local maximum stable set of G_3



Theorem (G. L. Nemhauser, L. E. Trotter Jr., Math. Progr., 1975)

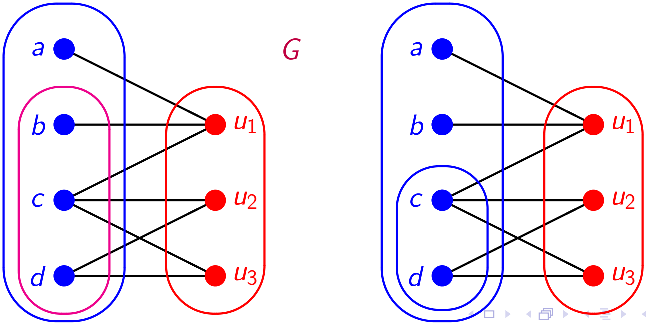
Every local maximum stable set of a graph is a subset of a maximum stable set.

Example

$\{b, c, d\} \in \Psi(G)$ and $\{b, c, d\} \subset \{a, b, c, d\} \in \Omega(G)$.

$\{c, d\} \notin \Psi(G)$, but $\{c, d\} \subset \{a, b, c, d\} \in \Omega(G)$.

Hence, the converse of the above theorem is not true.



Definition (H. Witney, 1935)

A **matroid** *is a pair* $(V, \mathcal{F})$*, where* $\mathcal{F} \subseteq 2^V$ *is a non-empty set system satisfying the following conditions:*

- **Hereditary property:** *if* $X \in \mathcal{F}$ *and* $Y \subset X$*, then* $Y \in \mathcal{F}$*.*
- *Exchange property:* for $X, Y \in \mathcal{F}, |X| = |Y| + 1$*,*
  there is an $x \in X - Y$ *such that* $Y \cup \{x\} \in \mathcal{F}$*.*

Definition (B. Korte, L. Lovasz, R. Schrader, 1991)

A **greedoid** *is a pair* $(V, \mathcal{F})$*, where* $\mathcal{F} \subseteq 2^V$ *is a non-empty set system satisfying the following conditions:*

- **Accessibility property:** *for every non-empty* $X \in \mathcal{F}$*,*
  there is an $x \in X$ *such that* $X - \{x\} \in \mathcal{F}$*;*
- *Exchange property:* for $X, Y \in \mathcal{F}, |X| = |Y| + 1$*,*
  there is an $x \in X - Y$ *such that* $Y \cup \{x\} \in \mathcal{F}$*.*

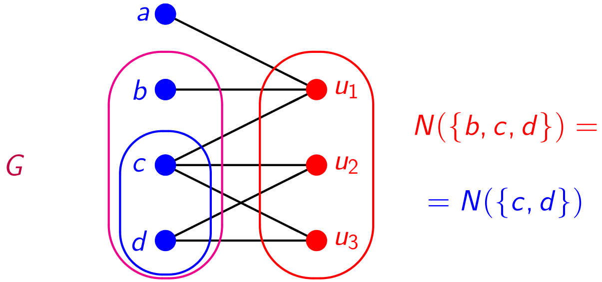
Remark : Clearly, every matroid is a greedoid, as well.

Is the family of local maximum stable sets hereditary?

- **Question:** is the family $\Psi(G)$ of all **loc max st** sets **hereditary**?

  i.e., if $B \subset A$ and $A \in \Psi(G)$, then $B \in \Psi(G)$?

- The answer is (generally) **NO**.
- **Example:** $\{c,d\} \subset \{b,c,d\} \in \Psi(G)$, but $\{c,d\} \notin \Psi(G)$.

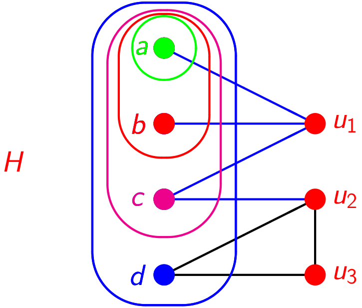


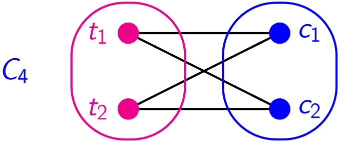
The family of local maximum stable sets and accessibility

- **Question:** does $\Psi(G)$ satisfies the accessibility property?

  i.e., if $\varnothing \neq A \in \Psi(G)$, then $A - \{a\} \in \Psi(G)$ for some $a \in A$?
- The answer is (generally) **NO**.
- $\Psi(H)$ has the accessibility property;

  e.g., $\{a\} \subset \{a, b\} \subset \{a, b, c\} \subset \{a, b, c, d\}$ and all are in $\Psi(H)$
- **No** accessibility in $\Psi(C_4)$.





$\{t_1, t_2\} \in \Psi(C_4)$

$\{t_1\}, \{t_2\} \notin \Psi(C_4)$

Greedoids on vertex sets of forests

Theorem (Levit and Mandrescu, Discrete Applied Mathematics, 2002)

If G is a forest, then $\Psi(G)$ forms a greedoid on its vertex set.

Example

- $\Psi(T)$ consists of the following sets:

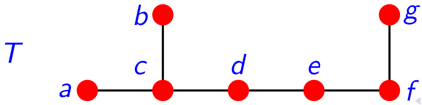
  $\emptyset, \{a\}, \{b\}, \{g\}$ and

  $\{a,b\}, \{a,d\}, \{b,d\}, \{e,g\}, \{a,g\}, \{b,g\}$ and

  $\{a,b,d\}, \{a,b,g\}, \{b,e,g\}$ and

  $\{a,b,d,f\}, \{a,b,d,g\}, \{a,b,e,g\}$.

- $\Psi(T)$ forms a greedoid on the vertex set $V = \{a,b,c,d,e,f,g\}$.



Proposition (Levit and Mandrescu, LNCS 5165, 2008)

If Y is the **clique corona** *of G and $\{H_1, H_2, ..., H_n\}$, then $\Psi(Y)$ is a greedoid, for any graph G.*

Examples

- Both $\Psi(Y_1)$ and $\Psi(Y_2)$ are greedoids.
- $\Psi(G)$ is **not** a greedoid.

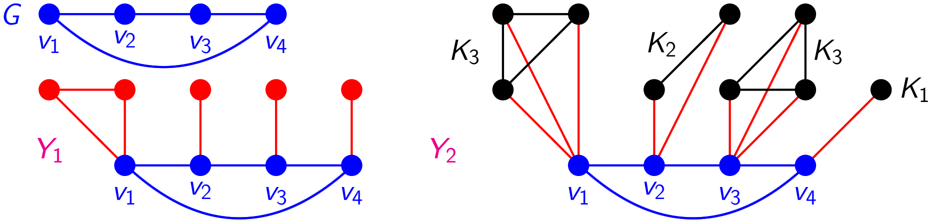


Figure: The graphs $Y_1 = G \circ \{K_2, K_1, K_1, K_1\}$ and $Y_2 = G \circ \{K_3, K_2, K_3, K_1\}$.

Theorem (Levit and Mandrescu, Discrete Applied Mathematics, 2010)

Let $H_1, H_2, ..., H_n$ be non-empty graphs and $Y = G \circ \{H_1, H_2, ..., H_n\}$.

Then $\Psi(Y)$ is a greedoid **iff** *every $\Psi(H_i), i = 1, 2, ..., n$ is a greedoid.*

Examples

- Both $\Psi(Y_1)$ and $\Psi(Y_2)$ are greedoids.
- $\Psi(G)$ is **not** a greedoid.

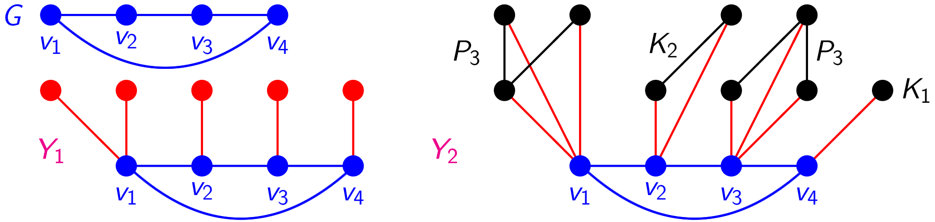


Figure: The graphs $Y_1 = G \circ \{2K_1, K_1, K_1, K_1\}$ and $Y_2 = G \circ \{P_3, K_2, P_3, K_1\}$.

Some conclusions

We saw clique corona graphs involved in:

- characterizations of (very) well-covered graphs of girth > 6
- building families of well-covered graphs of any girth
- building families of graphs whose independence polynomials enjoy various properties
- building families of graphs whose local maximum stable sets form greedoids.

Gracias por su atención !
Obrigado pela sua atenção !
Thank you for your attention !

Some old open problems

- Characterize graphs whose independence polynomials are unimodal.
- Characterize graphs whose independence polynomials have only real roots.
- Characterize polynomials that are independence polynomials of some graphs.
- Characterize unimodal polynomials that are independence polynomials of some graphs.

Thank you for your attention !

Conjecture

The independence polynomial of every 1-well-covered graph is unimodal.

Questions and facts on $I(G;x) = s_0 + s_1 x + s_2 x^2 + ... + s_{\alpha(G)} x^{\alpha(G)}$

- Is $I(G;x)$ **unimodal**, i.e., is there some $k \in \{0, 1, ..., n\}$,
  such that $s_0 \leq ... \leq s_{k-1} \leq s_k \geq s_{k+1} \geq ... \geq s_{\alpha(G)}$?

Theorem (Y. Alavi, P. J. Malde, A. J. Schwenk and P. Erdös, 1987)

For every permutation π of $\{1, 2, ..., \alpha\}$, there is a graph G with $\alpha(G) = \alpha$, such that $s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)}$.

- **Conjecture:** $I(T;x)$ is unimodal for any tree T (Alavi et al., 1987).
- Some **"Pro's"**: (Levit and Mandrescu, 2007)
  (i) $s_{\lceil (2\alpha-1)/3 \rceil} \geq ... \geq s_{\alpha-1} \geq s_\alpha$ hold for any tree T with $\alpha(T) = \alpha$;
  (ii) If the tree T has a perfect matching consisting of only
  pendant edges and $\alpha(T) = \alpha$, then

Questions and facts on $I(G;x) = s_0 + s_1 x + s_2 x^2 + ... + s_{\alpha(G)} x^{\alpha(G)}$

Theorem (Alavi, Malde, Schwenk and Erdös, 1987)

For every permutation π of $\{1, 2, ..., \alpha\}$, there is a graph G with $\alpha(G) = \alpha$, such that $s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)}$.

- **Conjecture:** $I(T;x)$ is unimodal for any tree T (Alavi et al., 1987).
- **Ex-Conjecture:** $I(G;x)$ is unimodal for every bipartite graph G. (Levit and Mandrescu, 2006).
- D. Galvin, 2012
  $I(G;x)$ is unimodal for almost every bipartite graph G.
- A. Bhattacharyya and J. Kahn, 2013
  There is a bipartite graph G whose $I(G;x)$ is not unimodal.

Questions and facts on $I(G;x) = s_0 + s_1x + s_2x^2 + ... + s_{\alpha(G)}x^{\alpha(G)}$

- Is $I(G;x)$ **log-concave**, i.e., $s_i^2 \geq s_{i-1} \cdot s_{i+1}$ for $1 \leq i < \alpha(G)$?
- Folklore: If $I(G;x)$ is log-concave, then $I(G;x)$ is also unimodal.
- **Some results:**

(i) $I(G;x)$ is log-concave, whenever all its roots are real (I. Newton).

(ii) $I(G;x)$ is log-concave for any claw-free graph G (the "claw" is $K_{1,3}$); (Y. O. Hamidoune, 1990)

(iii) $I(G \circ K_1;x)$ is log-concave for every graph G with $\alpha(G) \leq 3$ (Levit and Mandrescu, 2004)

(iv) $I(K_{1,n} \circ K_1;x)$ is log-concave for every n-star $K_{1,n}$ (Levit and Mandrescu, 2004)

Questions and facts on $I(G;x) = s_0 + s_1x + s_2x^2 + ... + s_{\alpha(G)}x^{\alpha(G)}$

- Are **all the roots** of $I(G;x)$ real?
- **Some results:**

(i) a root of smallest modulus of $I(G;x)$ is real
(J. Brown, K. Dilcher, R. Nowakowski, J. Algebr. Comb. 2000).

(ii) if $L(G)$ is the line graph of G, then $I(L(G);x)$ has all its roots real, since $I(L(G);x) = M(G;x)$ and $M(G;x)$ has only real roots.

(iii) if G is claw-free, then $I(G;x)$ has all its roots real
(M. Chudnovsky, P. Seymour, J. Comb. Th. 2007).

(iv) $I(G;x)$ has only real roots **iff** $I(G \circ K_1;x)$ has all the roots real,
(Levit and Mandrescu, DAM 2008).

(v) All the real roots of $I(G \circ K_p;x)$ belong to $\left[-\frac{1}{p}, -\frac{1}{n(p+1)}\right)$,
where $n = |V(G)|$.
(Mandrescu, Graphs and Combin. 2009).

Questions and facts on $I(G;x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_{\alpha(G)} x^{\alpha(G)}$

- Are there **closed formulae** for $I(G;x)$?
- **Some results:**

  (i) If P_n denotes the chordless path on $n \geq 3$, then

  $$I(P_n;x) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-j}{j} \cdot x^j.$$

  (J. L. Arocha, 1984).

  (ii) If C_n denotes the chordless cycle on $n \geq 3$, then

  $$I(C_n;x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \cdot \binom{n-j}{j} \cdot x^j$$

  (F. Harary and I. Gutman, 1983).

  (iii) if S_n denotes the graph obtained from $K_{1,n}$ by appending a single pendant edge to each vertex of $K_{1,n}$, then

  $$I(S_n;x) = (1+x) \cdot \sum_{k=0}^{n} \left[\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k,$$

  (Levit and Mandrescu, 2008)

Zykov sum of graphs and its independence polynomial

Definition (A. A. Zykov, 1990)

The **Zykov sum** *of two disjoint graphs G and H is the graph $G + H$ obtained by joining each vertex of G with each vertex of H by an edge.*

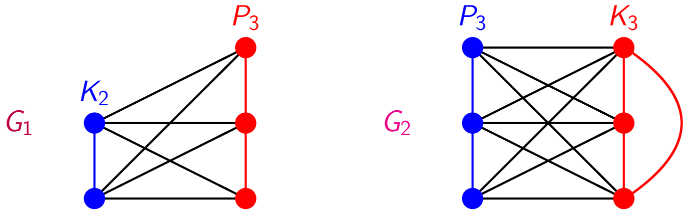


Figure: $G_1 = K_2 + P_3$ and $G_2 = P_3 + K_3$.

Fact (Gutman and Harary, '83): $I(G + H; x) = I(G; x) + I(H; x) - 1$

- $I(P_3 + K_3; x) = (1 + 3x + x^2) + (1 + 3x) - 1 = 1 + 6x + x^2$.

Non-isomorphic graphs with the same indep poly

Question

Are there **non-isomorphic** *graphs* G_1, G_2 *with* $I(G_1;x) = I(G_2;x)$?

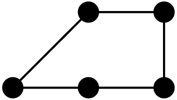
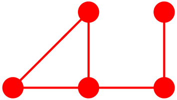
G_1  G_2 

Figure: $I(G_1;x) = I(G_2;x) = 1 + 5x + 5x^2$.

Example (K. Dohmen, A. Pönitz and P. Tittmann, DMTCS 2003)

There are **non-isomorphic trees** *with the same independence polynomial.*

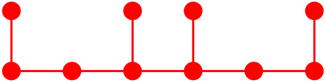
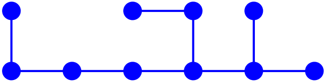
T_1  T_2 

Figure: $I(T_1;x) = I(T_2;x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6$.