A linear algorithm for the k-tuple chromatic number of partner limited graphs

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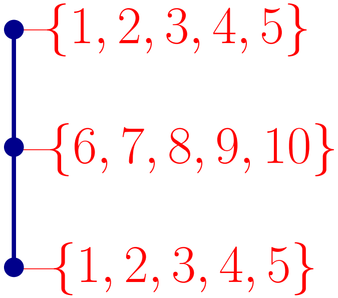
The k-tuple coloring problem

Definition

In the *k-tuple coloring problem*, we aim to assign sets of colors of size k to the vertices of a graph G, so that the sets which belong to adjacent vertices of G have empty intersection, and the total number of colors used is minimum. This minimum number of colors is called the *k-tuple chromatic number* χ_k.

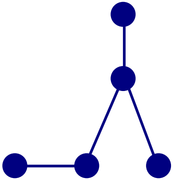


A 5-tuple coloring of G

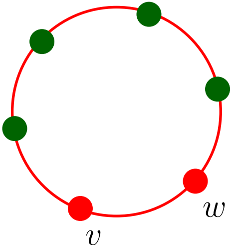
P_4-tidy graphs

Definition

A graph is P_4-*tidy* if there exists for every induced P_4 A at most one vertex $v \in G - A$ such that $A \cup \{v\}$ induces at least two P_4s in G.



(a) A P_4-tidy graph



(b) A non P_4-tidy graph

Main result

Theorem

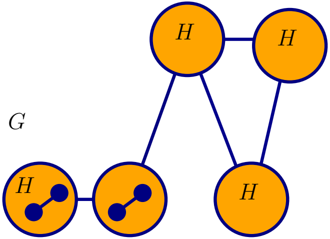
The k-tuple chromatic number of a P_4-tidy graph G can be computed in linear time.

Lexicographic products on graphs

Definition

The *lexicographic product* of the graph G by the graph H is the graph $G \circ H$ with vertex set $V(G) \times V(H)$ and edge set:

$$E(G \circ H) = \{(a,x)(b,y) \mid ab \in E(G), \text{ or } a = b \text{ and } xy \in E(H)\}$$

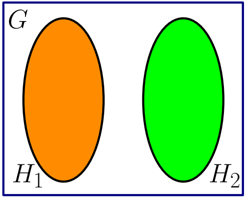


Remark

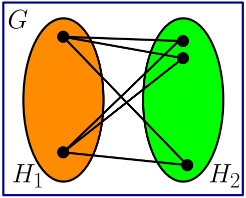
$\chi_k(G) = \chi(G \circ K_k)$.

Theorem [Olariu et. al]

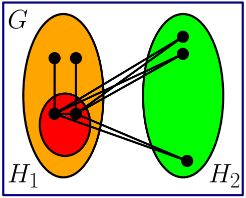
Every graph can be constructed by repeated application of three binary operations $\{\cup, \vee, \oplus\}$, starting with a family F of elemental graphs.



(c) Union $G = H_1 \cup H_2$

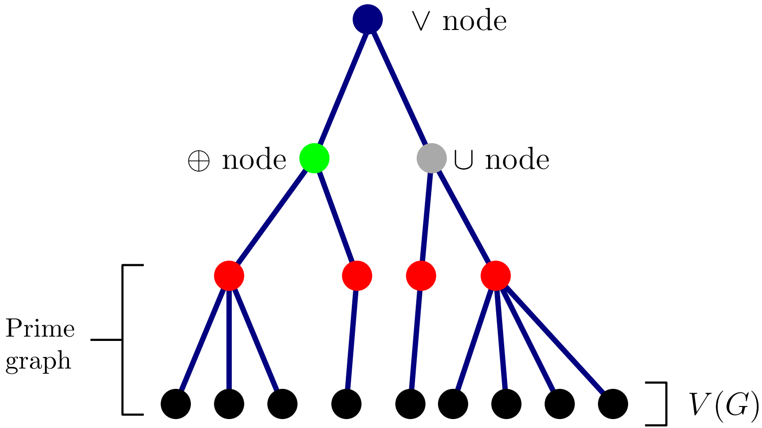


(d) Join $G = H_1 \vee H_2$



(e) P_4-join $G = H_1 \oplus H_2$

Primeval decomposition tree of G



Decompositions of lexicographic products

Proposition (folklore)

Let G_1, G_2 and H be graphs. Then

i.- $(G_1 \cup G_2) \circ H = (G_1 \circ H) \cup (G_2 \circ H)$.

ii.- $(G_1 \vee G_2) \circ H = (G_1 \circ H) \vee (G_2 \circ H)$.

Proposition

Let G_2 be a graph, G_1 a graph with a (not too horrible) technical condition, and H be a graph. Then $(G_1 \oplus G_2) \circ H = (G_1 \circ H) \oplus (G_2 \circ H)$.

This leads us to

Proposition

Let G and H be graphs. The decomposition tree of $G \circ H$ can be obtained from the decomposition tree of G.

Relating colorings with decompositions

Theorem (folklore)

If G is the trivial graph, then $\chi(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,

i. $\chi(G_1 \cup G_2) = max\{\chi(G_1), \chi(G_2)\}$

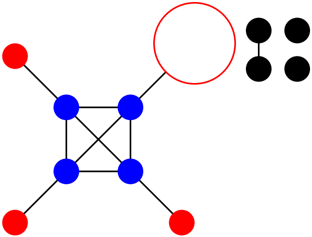
ii. $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.

Lemma

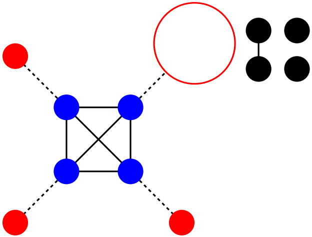
Let G_1 be a graph that can appear as first operand of a $\oplus$ operation in the decomposition of $G \circ K_k$, with G P_4-tidy, and G_2 be a graph. Then $\chi(G_1 \oplus G_2) = \chi(G_1) + \chi(G_2)$.

Elemental graphs

- K_k
- $P_5 \circ K_k$
- $\overline{P_5} \circ K_k$
- $C_5 \circ K_k$
- $Quasi - starfishes \circ K_k$
- $Quasi - urchins \circ K_k$



(f) A quasi urchin



(g) A quasi starfish

Main result

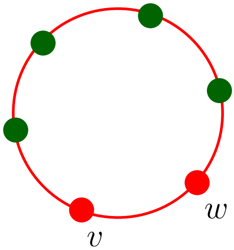
Theorem

The k-tuple chromatic number of a P_4-tidy graph G can be computed in linear time.

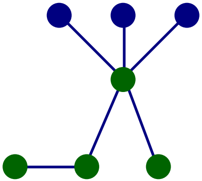
Partner limited graphs

Definition

A graph is *partner limited* if there exists for every induced P_4 A at most two vertices $v \in G - A$ such that $A \cup \{v\}$ induces at least two P_4s in G.



(h) A partner limited graph



(i) A non partner limited graph

Main result

Theorem

The k-tuple chromatic number of a partner limited graph G can be computed in linear time.

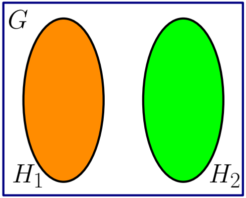
Modular decomposition

Definition

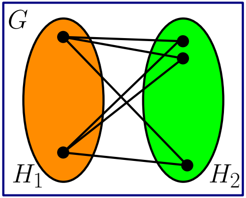
Let $G = (V, E)$ be a graph. A set $M \subseteq V$ of vertices is called a *module* if every vertex in $V \setminus M$ is either adjacent to all vertices in M , or to none of them.

Theorem (Gallai et al.)

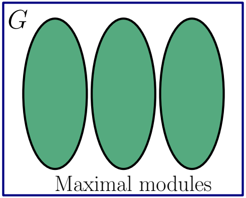
Every graph can be constructed by repeated application of three operations ($\cup, \vee, \uplus$), starting with a family F of *prime* modules. This is called the *modular decomposition tree* $T(G)$ of G.



(j) $G = H_1 \cup H_2$

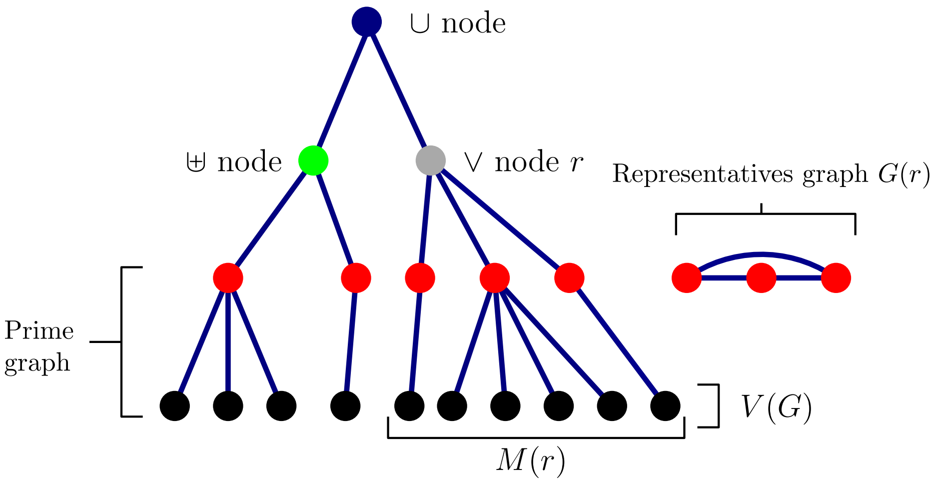


(k) $G = H_1 \vee H_2$



(l) $G = \uplus(H_1, H_2, \ldots)$

Modular decomposition tree

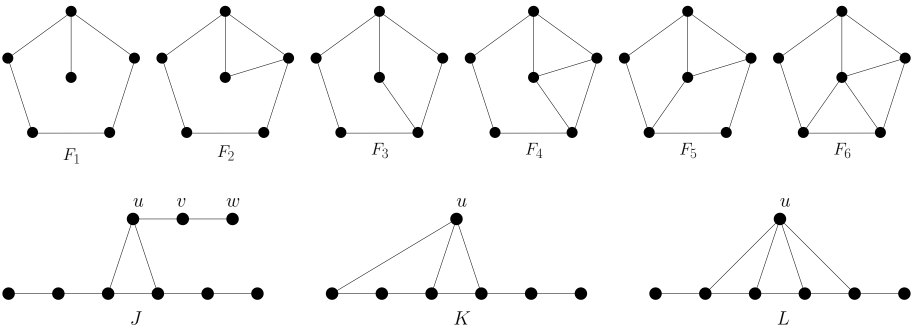


Decomposition theorem for PL-graphs

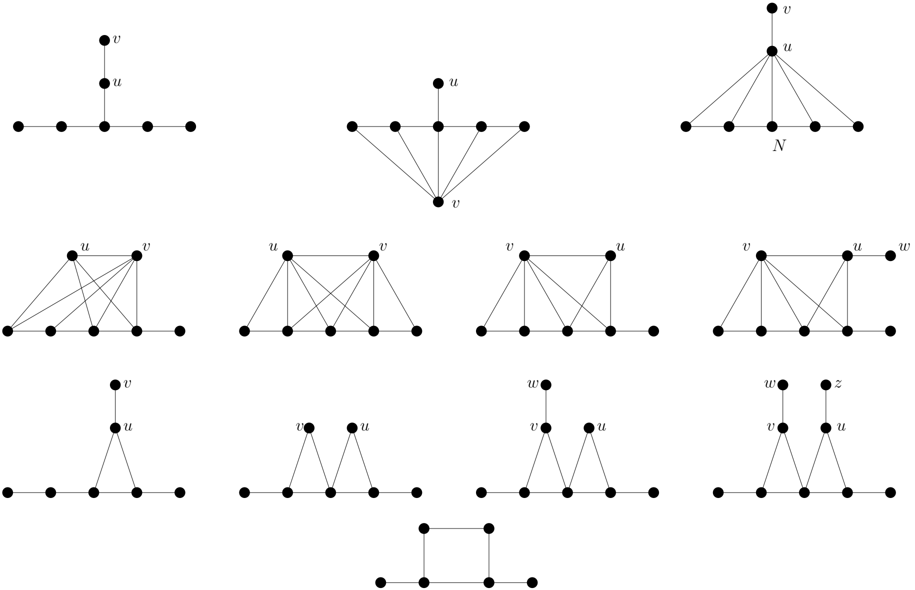
Theorem [Roussel et. al. 1999]

A graph $G = (V, E)$ is a PL-graph if and only if for each vertex $v \in T(G)$ at least one of the following statements is valid:

1. $M(v)$ is a module isomorphic to a graph below (denote this set by ZOO).

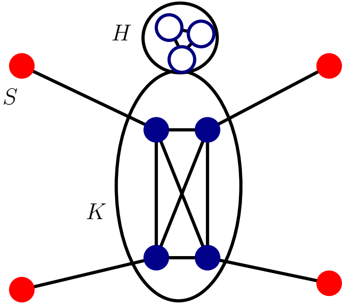


Decomposition theorem for PL-graphs



Decomposition theorem for PL-graphs

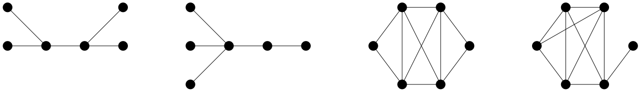
2. In G or $\overline{G}$, $M(v)$ is isomorphic to a cycle $C_n, n \geq 6$.
3. G or $\overline{G}$ is disconnected.
4. In G or $\overline{G}$, $G(v)$ is a well labelled spider.

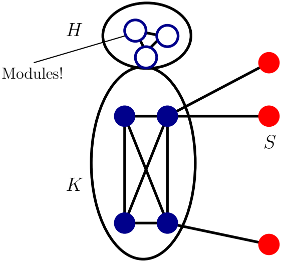


A spider G is a kind of split graph $G = (H, K, S)$. A well labelled spider is a slightly modified spider.

Decomposition theorem for PL-graphs

5. $G(v)$ is a split graph (H, K, S) with no $H_1, H_2, \overline{H_1}, \overline{H_2}$ and such that for every $v \in K \cup S$ we have $|M(v)| = 1$ (i.e. v is a leaf of $T(G)$).





Outline of proof

1. We can also deduce in this case the decomposition tree for $G \circ K_k$, with G PL-graph.
2. In G or $\overline{G}$, $M(v)$ is isomorphic to a cycle $C_n, n \geq 6$.

Theorem[Bonomo et al.]

Let C_r be a cycle on $r = 2t + 1$ vertices. Then, $\chi_k(C_r) = max\{2k, \lceil \frac{rk}{t} \rceil\}$ and a k-tuple coloring of C_r with $\chi_k(C_r)$ colors can be obtained in $O(r)$ time.

Proposition

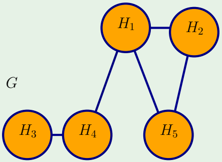
Let $C_n, n \geq 5$ be and odd cycle. Then $\chi_k(\overline{C_n}) = \lceil \frac{kn}{2} \rceil$.

Outline of proof

4. In G or $\overline{G}$, $M(v)$ is a well labelled spider.
5. $G(v)$ is a split graph (H, K, S) with no $H_1, H_2, \overline{H_1}, \overline{H_2}$ and such that for every $v \in K \cup S$ we have $|M(v)| = 1$.

Definition

Generalized lexicographical product $G \circ \{H_1, H_2, \ldots, H_n\}$



Proposition

If G is a split graph, we have an expression for $\chi(G \circ \{H_1, \ldots, H_n\})$ based on $\chi(H_i), \ldots, \chi(H_n)$.

Main result

Theorem

The k-tuple chromatic number of a partner limited graph G can be computed in linear time.

Questions?
Muchas gracias!