Critical Ideals of Digraphs

Carlos A. Alfaro, Carlos E. Valencia, Adrián Vázquez
Banxico, CINVESTAV-IPN, UNAQ

VII Latin American Workshop on Cliques in Graphs,
La Plata, Argentina, November 8–11, 2016
Laplacian Matrix

Definition

Let $G = (V, E)$ be a graph, the Laplacian matrix $L(G)$ of $G$ is the matrix with rows and columns indexed by the vertices of $G$ given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of $u$, and $m_{uv}$ denote the number of edges from $u$ to $v$.

$$L(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$
Definition

By considering the Laplacian matrix $L(G)$ as a linear operator on $\mathbb{Z}^V$, the critical group $K(G)$ of $G$ is the torsion part of the cokernel of $L(G)$.

$$coker(L(G)) = \mathbb{Z}^V/\text{Im}L(G) = \mathbb{Z} \oplus K(G).$$
Invarian factors

\[ K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}}, \]

where \( f_i \geq 0 \) and \( f_i \mid f_j \) for all \( i \leq j \).
Invariant factors

\[ K(G) \cong \mathbb{Z} f_1 \oplus \mathbb{Z} f_2 \oplus \cdots \oplus \mathbb{Z} f_{n-1}, \]

where \( f_i \geq 0 \) and \( f_i \mid f_j \) for all \( i \leq j \).

The \( f_1, f_2, \ldots, f_{n-1} \) are called invariant factors.
Invarian factors

\[ K(G) \cong \mathbb{Z} f_1 \oplus \mathbb{Z} f_2 \oplus \cdots \oplus \mathbb{Z} f_{n-1}, \]

where \( f_i \geq 0 \) and \( f_i | f_j \) for all \( i \leq j \).

The \( f_1, f_2, \ldots, f_{n-1} \) are called invariant factors.

Let \( \Delta_i(G) \) be the g.c.d of the \( i \)-minors of \( L(G) \). Then

\[ f_i = \frac{\Delta_i(G)}{\Delta_{i-1}(G)}, \]

where \( \Delta_0(G) = 1 \).
The family $\mathcal{G}_i$

**Definition**

Let $f_1(G)$ be the number of invariant factors of $L(G)$ equal to 1.
The family $\mathcal{G}_i$

**Definition**
Let $f_1(G)$ be the number of invariant factors of $L(G)$ equal to 1.

**Definition**
Let $\mathcal{G}_i$ be the family of simple connected graphs with $f_1(G) = i$. 

Example
The following graph belongs to $\mathcal{G}_2$.

![Graph](image-url)
The family $\mathcal{G}_i$

**Definition**
Let $f_1(G)$ be the number of invariant factors of $L(G)$ equal to 1.

**Definition**
Let $\mathcal{G}_i$ be the family of simple connected graphs with $f_1(G) = i$.

**Example**
The following graph belongs to $\mathcal{G}_2$.

$L(G) \sim \text{diag}(1, 1, 3, 3, 0)$
Motivation: Critical group

Critical ideals

Some properties

- Given simple graph, there is an homeomorphic graph with cyclic critical group (Chen & Ye, 2008).
- There are graph operations that preserve cyclicity of the critical group (Krepkiy, 2014).
- The trees are the simple connected graphs with $n$ vertices and $n - 1$ invariant factors equal to 1.
Motivation: Critical group

Critical ideals

Question

How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n - 2$ or $n - 1$?

Conjecture (D. Wagner, 2001)
Almost every connected simple graph has a cyclic critical group.

Theorem (M. Wood, 2014)
The probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3) - \frac{1}{\zeta(5)} - \frac{1}{\zeta(7)} - \frac{1}{\zeta(9)} - \frac{1}{\zeta(11)} - \cdots \approx 0.7935212$$

where $\zeta$ is the Riemann zeta function.
Question
How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n - 2$ or $n - 1$?

Conjeteru (D. Wagner, 2001)
Almost every connected simple graph has a cyclic critical group.
Question
How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n - 2$ or $n - 1$?

Conjecture (D. Wagner, 2001)
Almost every connected simple graph has a cyclic critical group.

Theorem (M. Wood, 2014)
The probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1} \cdots \approx 0.7935212$$

where $\zeta$ is the Riemann zeta function.
Motivation: Critical group

Critical ideals

Graphs with one invariant factor equal to 1

On the other hand...

Theorem (Lorenzini, 1991)

Let $G$ be a simple connected graph. Then the following statements are equivalent:

I. $G \in G_1$,

II. $G$ is $P_3$-free, where $P_n$ denote the path with $n$ vertices,

III. $G$ is a complete graph.

Question

What can we say about $G_1$?
Graphs with one invariant factor equal to 1

On the other hand...

**Question**

What can we say about $G_1$?
On the other hand...

**Question**

What can we say about $G_1$?

**Theorem (Lorenzini, 1991)**

Let $G$ be a simple connected graph. Then the following statements are equivalent:

I. $G \in G_1$,

II. $G$ is $P_3$-free, where $P_n$ denote the path with $n$ vertices,

III. $G$ is a complete graph.
Motivation: Critical group

Critical ideals

Digraphs with one invariant factor equal to 1

**Question**

What can we say about digraphs?
Question
What can we say about digraphs?

Theorem (Alfaro, Valencia & Vázquez)
The critical group of a connected digraph has exactly 1 invariant factor equal to 1 if and only if it is isomorphic to the digraph $\Lambda_{n_1, n_2, n_3}$ where $n_1, n_2, n_3$ satisfy one of the following conditions:

- $n_1, n_2, n_3 \geq 1$,
- $n_1 = n_2 = 1, n_3 = 0$,
- $n_1 = n_3 = 1, n_2 = 0$,
- $n_2 = n_3 = 1, n_1 = 0$,
- $n_1 \geq 1, n_2 \geq 2, n_3 = 0$,
- $n_1 = 0, n_2 \geq 2, n_3 = 0$,
- $n_1 = 0, n_2 = 1, n_3 \geq 2$,
- $n_1 = 0, n_2 \geq 2, n_3 \geq 1$,
- $n_1 = 1, n_2 = 0, n_3 \geq 2$,
- $n_1 \geq 2, n_2 = 0, n_3 \geq 2$. 

[Diagram of a digraph with nodes $n_1, n_2, n_3$ and directed edges connecting them]
**The critical ideals of digraph**

**Definition**

Given a digraph $D = (V, A)$ and a set of indeterminates $X_D = \{x_u : u \in V\}$, the generalized Laplacian matrix $L(D, X_D)$ of $D$ is the matrix given by

$$L(D, X_D)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise}. \end{cases}$$

![Graph Diagram]

$$L(D, X_D) = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ -1 & x_2 & -1 & 0 & 0 & -1 \\ 0 & -1 & x_3 & -1 & 0 & -1 \\ 0 & 0 & -1 & x_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & x_5 & 0 \\ 0 & -1 & -1 & 0 & 0 & x_6 \end{bmatrix}$$
For all $1 \leq i \leq |V(D)|$, the $i$-th critical ideal of $D$ is the determinantal ideal given by

$$I_i(D, X_D) = \langle \{ m : m \text{ is an } i\text{-minor of } L(D, X_D) \} \rangle \subseteq \mathbb{Z}[X_D].$$
Motivation: Critical group

Critical ideals

**Definition**

For all $1 \leq i \leq |V(D)|$, the $i$-th critical ideal of $D$ is the determinantal ideal given by

$$I_i(D, X_D) = \left\langle \{m : m \text{ is an } i\text{-minor of } L(D, X_D)\} \right\rangle \subseteq \mathbb{Z}[X_D].$$

**Definition**

We say that a critical ideal is **trivial** when it is equal to $\langle 1 \rangle$. 


Definition
For all $1 \leq i \leq |V(D)|$, the $i$-th critical ideal of $D$ is the determinantal ideal given by

$$I_i(D, X_D) = \langle \{m : m \text{ is an } i\text{-minor of } L(D, X_D)\} \rangle \subseteq \mathbb{Z}[X_D].$$

Definition
We say that a critical ideal is **trivial** when it is equal to $\langle 1 \rangle$.

Definition
The *algebraic co-rank* $\gamma(D)$ of a digraph $D$ is the number of trivial critical ideals of $D$. 
Remark

- $G_k$ is not closed under induced subgraphs. For instance, $c(S_3)$ belongs to $G_2$, but $S_3$ belongs to $G_3$. Similarly, $K_6 \setminus \{2P_2\}$ belongs to $G_3$ meanwhile $K_5 \setminus \{2P_2\}$ belongs to $G_2$.

- If $H$ is an induced subgraph of $G$, it is not always true that $K(H) \sqsubseteq K(G)$. For example, $K(K_4) \simeq \mathbb{Z}_4^2 \not\cong K(K_5) \simeq \mathbb{Z}_5^3$. 
Remark

- $G_k$ is not closed under induced subgraphs. For instance, $c(S_3)$ belongs to $G_2$, but $S_3$ belongs to $G_3$. Similarly, $K_6 \setminus \{2P_2\}$ belongs to $G_3$ meanwhile $K_5 \setminus \{2P_2\}$ belongs to $G_2$.

- If $H$ is an induced subgraph of $G$, it is not always true that $K(H) \subseteq K(G)$. For example, $K(K_4) \cong \mathbb{Z}_4^2 \not\cong K(K_5) \cong \mathbb{Z}_5^3$.

Remark

If $H$ is an induced subdigraph of $G$, then $l_i(H, X_H) \subseteq l_i(G, X_G)$ for all $i \leq |V(H)|$.

$\gamma(H) \leq \gamma(G)$. 
Definition

\[ \Gamma_{\leq k} = \{ D : D \text{ is a simple connected digraph with } \gamma(D) \leq k \} . \]
**Definition**

\[ \Gamma \leq k = \{ D : D \text{ is a simple connected digraph with } \gamma(D) \leq k \} . \]

**Remark**

\( \Gamma \leq k \) is closed under induced subdigraphs.
Definition

A digraph $D$ is forbidden for $\Gamma \leq k$ when $\gamma(D) \geq k + 1$. 
A digraph $D$ is forbidden for $\Gamma \leq k$ when $\gamma(D) \geq k + 1$.

Let $\text{Forb}(\Gamma \leq k)$ be the set of minimal (under induced subdigraphs property) forbidden digraphs for $\Gamma \leq k$. 
Definition
A digraph $D$ is forbidden for $\Gamma_{\leq k}$ when $\gamma(D) \geq k + 1$.

Definition
Let Forb($\Gamma_{\leq k}$) be the set of minimal (under induced subdigraphs property) forbidden digraphs for $\Gamma_{\leq k}$.

Remark
$D \in \Gamma_{\leq k}$ if and only if $D$ is Forb($\Gamma_{\leq k}$)-free.
Theorem (Alfaro, Valencia, Vazquez)

Let $D$ be a connected digraph. Then the following statements are equivalent:

(i) $D \in \Gamma_{\leq 1},$

(ii) $D$ is $\mathfrak{F}$-free.

(iii) $D$ is isomorphic to $\Lambda_{n_1,n_2,n_3}.$
Motivation: Critical group

Critical ideals

**Figura:** The \( \gamma \)-critical digraphs with 3 and 4 vertices that have algebraic co-rank equal to 2.
Recalling: Approach to solve this problem

1. Find the family $F$ of induced forbidden subgraphs for $\Gamma \leq i$.
2. Determine the structure of the $F$-free graphs.
3. Compute the critical ideals of the $F$-free graphs.
4. Compute the critical group of the $F$-free graphs.
Recalling: Approach to solve this problem

1. Find the family $\mathcal{F}$ of induced forbidden subgraphs for $\Gamma_{\leq i}$. 
Recalling: Approach to solve this problem

1. Find the family $\mathcal{F}$ of induced forbidden subgraphs for $\Gamma_{\leq i}$.
2. Determine the structure of the $\mathcal{F}$-free graphs.
Recalling: Approach to solve this problem

1. Find the family $\mathcal{F}$ of induced forbidden subgraphs for $\Gamma_{\leq i}$.
2. Determine the structure of the $\mathcal{F}$-free graphs.
3. Compute the critical ideals of the $\mathcal{F}$-free graphs.
Recalling: Approach to solve this problem

1. Find the family $\mathcal{F}$ of induced forbidden subgraphs for $\Gamma_{\leq i}$.
2. Determine the structure of the $\mathcal{F}$-free graphs.
3. Compute the critical ideals of the $\mathcal{F}$-free graphs.
4. Compute the critical group of the $\mathcal{F}$-free graphs.
Given a graph $G = (V, E)$ and $\delta \in \{0, 1, -1\}^{|V|}$, let

$$\mathcal{T}_\delta(G) = \{G^d : d \in \mathbb{Z}^{|V|} \text{ such that } \text{supp}(d) = \delta\},$$

where $G^d$ is the graph obtained by duplicating $d_v$ times the vertex $v$ when $d_v > 0$ and replicating $-d_v$ times the vertex $v$ when $d_v < 0$. The algebraic co-rank of any graph in $\mathcal{T}_\delta(G)$ is equal to the algebraic co-rank of $G^\delta$. 

Theorem (Alfaro, Corrales, Valencia, 2015)
Definition

Given a graph $G = (V, E)$ and $\delta \in \{0, 1, -1\}^{|V|}$, let

$$T_\delta(G) = \{ G^d : d \in \mathbb{Z}^{|V|} \text{ such that } \text{supp}(d) = \delta \},$$

where $G^d$ is the graph obtained by duplicating $d_v$ times the vertex $v$ when $d_v > 0$ and replicating $-d_v$ times the vertex $v$ when $d_v < 0$.

Theorem (Alfaro, Corrales, Valencia, 2015)

The algebraic co-rank of any graph in $T_\delta(G)$ is equal to the algebraic co-rank of $G^\delta$. 
Another approach to solve this problem

1. Find all possible values for \((G, \delta)\) that characterize \(\Gamma_{\leq k}\).
Another approach to solve this problem

1. Find all possible values for \((G, \delta)\) that characterize \(\Gamma \leq k\).
2. Compute the critical group of the graphs in \(\Gamma \leq k\).
Thank you!