Critical Ideals of Digraphs

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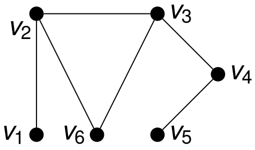
Laplacian Matrix

Definition

Let $G = (V, E)$ be a graph, the Laplacian matrix $L(G)$ of G is the matrix with rows and columns indexed by the vertices of G given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of u, and m_{uv} denote the number of edges from u to v.



$$L(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Definition

By considering the Laplacian matrix $L(G)$ as a linear operator on $\mathbb{Z}^V$, the critical group $K(G)$ of G is the torsion part of the cokernel of $L(G)$.

$$coker(L(G)) = \mathbb{Z}^V/\mathrm{Im}L(G) = \mathbb{Z} \oplus K(G).$$

Invarian factors

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}},$$

where $f_i \geq 0$ and $f_i \mid f_j$ for all $i \leq j$.

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Let $\Delta_i(G)$ be the g.c.d of the i-minors of $L(G)$. Then

$$f_i = \frac{\Delta_i(G)}{\Delta_{i-1}(G)},$$

where $\Delta_0(G) = 1$.

The family $\mathcal{G}_i$

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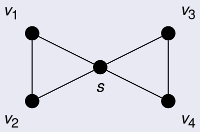
Example

The following graph belongs to $\mathcal{G}_2$.



$$L(G) \sim diag(1, 1, 3, 3, 0)$$

Some properties

- Given simple graph, there is an homeomorphic graph with cyclic critical group (Chen & Ye, 2008).
- There are graph operations that preserve cyclicality of the critical group (Krepkiy, 2014).
- The trees are the simple connected graphs with n vertices and $n-1$ invariant factors equal to 1.

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How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n-2$ or $n-1$?

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Almost every connected simple graph has a cyclic critical group.

Theorem (M. Wood, 2014)

The probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\cdots \approx 0{,}7935212$$

where ζ is the Riemann zeta function.

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On the other hand...

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What can we say about $\mathcal{G}_1$?

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Theorem (Lorenzini, 1991)

Let G be a simple connected graph. Then the following statements are equivalent:

I. $G \in \mathcal{G}_1$,
II. G *is* P_3*-free, where* P_n *denote the path with* n *vertices,*
III. G *is a complete graph.*

Digraphs with one invariant factor equal to 1

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Theorem (Alfaro, Valencia & Vázquez)

The critical group of a connected digraph has exactly 1 invariant factor equal to 1 if and only if it is isomorphic to the digraph Λ_{n_1,n_2,n_3} where n_1, n_2, n_3 satisfy one of the following conditions:

- $n_1, n_2, n_3 \geq 1$,
- $n_1, = n_2 = 1$, $n_3 = 0$,
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- $n_2, = n_3 = 1$, $n_1 = 0$,
- $n_1 \geq 1$, $n_2 \geq 2$, $n_3 = 0$,
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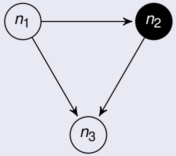
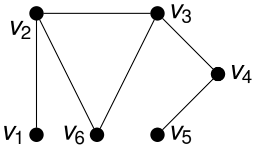


The critical ideals of digraph

Definition

Given a digraph $D = (V, A)$ and a set of indeterminates $X_D = \{x_u : u \in V\}$, the generalized Laplacian matrix $L(D, X_D)$ of D is the matrix given by

$$L(D, X_D)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise.} \end{cases}$$



$$L(D, X_D) = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ -1 & x_2 & -1 & 0 & 0 & -1 \\ 0 & -1 & x_3 & -1 & 0 & -1 \\ 0 & 0 & -1 & x_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & x_5 & 0 \\ 0 & -1 & -1 & 0 & 0 & x_6 \end{bmatrix}$$

Definition

For all $1 \leq i \leq |V(D)|$, the *i-th critical ideal* of D is the determinantal ideal given by

$$I_i(D, X_D) = \langle \{ m \ : \ m \text{ is an i-minor of } L(D, X_D) \} \rangle \subseteq \mathbb{Z}[X_D].$$

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Definition

The algebraic co-rank $\gamma(D)$ of a digraph D is the number of trivial critical ideals of D.

Remark

- $\mathcal{G}_k$ is not closed under induced subgraphs.
  For instance, $c(S_3)$ belongs to $\mathcal{G}_2$, but S_3 belongs to $\mathcal{G}_3$.
  Similarly, $K_6 \setminus \{2P_2\}$ belongs to $\mathcal{G}_3$ meanwhile $K_5 \setminus \{2P_2\}$ belongs to $\mathcal{G}_2$.
- If H is an induced subgraph of G, it is not always true that $K(H) \trianglelefteq K(G)$.
  For example, $K(K_4) \cong \mathbb{Z}_4^2 \ntrianglelefteq K(K_5) \cong \mathbb{Z}_5^3$.

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Remark

- If H is an induced subdigraph of G, then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $i \leq |V(H)|$.
- $\gamma(H) \leq \gamma(G)$.

Definition

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Let $\mathbf{Forb}(\Gamma_{\leq k})$ be the set of minimal (under induced subdigraphs property) forbidden digraphs for $\Gamma_{\leq k}$.

Remark

$D \in \Gamma_{\leq k}$ if and only if D is $\mathbf{Forb}(\Gamma_{\leq k})$-free.

Theorem (Alfaro, Valencia, Vazquez)

Let D be a connected digraph. Then the following statements are equivalent:

(I) $D \in \Gamma_{\leq 1}$,

(II) D *is* $\mathfrak{F}$*-free.*

(III) D *is isomorphic to* Λ_{n_1,n_2,n_3}.

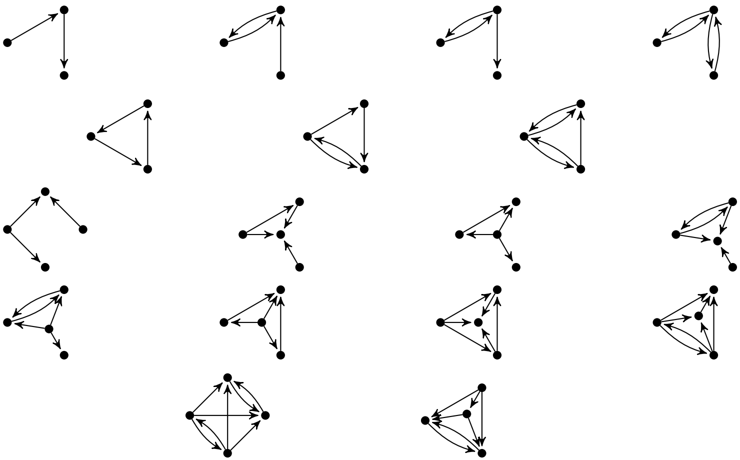

Figura: The γ-critical digraphs with 3 and 4 vertices that have algebraic co-rank equal to 2.

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2. Determine the structure of the $\mathcal{F}$-free graphs.
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4. Compute the critical group of the $\mathcal{F}$-free graphs.

Definition

Given a graph $G = (V, E)$ and $\delta \in \{0, 1, -1\}^{|V|}$, let

$$\mathcal{T}_\delta(G) = \{G^{\mathbf{d}} : \mathbf{d} \in \mathbb{Z}^{|V|} \text{ such that } \mathrm{supp}(\mathbf{d}) = \delta\},$$

where $G^{\mathbf{d}}$ is the graph obtained by duplicating $\mathbf{d}_v$ times the vertex v when $\mathbf{d}_v > 0$ and replicating $-\mathbf{d}_v$ times the vertex v when $\mathbf{d}_v < 0$.

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Theorem (Alfaro, Corrales, Valencia, 2015)

The algebraic co-rank of any graph in $\mathcal{T}_\delta(G)$ is equal to the algebraic co-rank of G^δ.

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1. Find all possible values for (G, δ) that characterize $\Gamma_{\leq k}$.

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1. Find all possible values for (G, δ) that characterize $\Gamma_{\leq k}$.
2. Compute the critical group of the graphs in $\Gamma_{\leq k}$.

Thank you!