Quantum Groups and Hopf Algebras

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Abstract

These notes correspond to a mini-course for graduate and undergraduate students given in the XVIII Latin American Algebra Colloquium in São Pedro, Brazil. Because of the lenght of the course, we intend only to give the basic ideas of the subject and to show, by means of examples, how quantum groups enter into the scene of the classification problem of finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero. The reader who is interested in this subject may have a look at the bibliography on quantum groups and Hopf algebras and references therein. We did not intend to give an exhaustive or comprehensive list of the references in these subjects, but just give some of them to serve as a guide.

1 Introduction

Quantum groups, introduced in 1986 by Drinfeld [Dr2], form a certain class of Hopf algebras. Up to date there is no rigorous, universally accepted definition, but it is generally agreed that this term includes certain deformations in one or more parameters of classical objects associated to algebraic groups, such as enveloping algebras of semisimple Lie algebras or algebras of regular functions on the corresponding algebraic groups. As one can relate algebraic groups with commutative Hopf algebras via group schemes, it is also agreed that the category of quantum groups should correspond to the opposite category of the category of Hopf algebras. This is why some authors define quantum groups as non-commutative and non-cocommutative Hopf algebras.

Hopf algebras were introduced in the 50's, and from the 60's they have been intensively studied. First in relation with algebraic groups and later as objects of self interest. One of the main open problems in the theory of Hopf algebras is the classification of Hopf algebras H of a fixed dimension over an algebraically closed field of characteristic zero. Up to now, very few general results are known and the classification is solved only if the dimension N of H is smaller or equal than 19, if N can be factorized in a simple way, *i.e.* $N = p, p^2, 2p^2$ with p a prime number, or if the Hopf algebra has additional properties such as semisimplicity or pointness. It turns out that there is a deep relation between semisimple Hopf algebras and group theory, and between pointed Hopf algebras and Lie theory. Since we do not assume some knowledge on Lie theory, we shall not talk about this relation. Nevertheless, it can be traced back from the examples coming from quantum groups.

One of the obstructions in solving the classification problem is the lack of enough examples. Hence, it is necessary to find new families of Hopf algebras. From the very beginning, this role was played by quantum groups. They consists of a large family with different structural properties and were used with profit to solve the classification problem for fixed dimensions.

After introducing Hopf algebras, together with some basic examples, we give in Section 3 the definition of the simpliest quantum groups $\mathcal{O}_q(SL_2(\Bbbk))$ and $U_q(\mathfrak{sl}_2)(\Bbbk)$ over an algebraically closed field $\Bbbk$ of characteristic zero. If q is a primitive ℓ-th root of unity of $\Bbbk$, then one can define the *small quantum group* $\mathbf{u}_q(\mathfrak{sl}_2)(\Bbbk)$, which is a finite-dimensional quotient of $U_q(\mathfrak{sl}_2)(\Bbbk)$ of dimension ℓ^3.

Finally in Section 4, we show how these small quantum groups and variation of them enter into the scene of the classification problem of Hopf algebras of dimension p^2 and pointed Hopf algebras of dimension p^3.

2 Hopf algebras

One of the major threads running through this subject has it roots in a philosophy proposed by Grothendieck, which states that one should study objects by means of the functions on them. This allows to relate algebraic objects to geometric objects and *vice versa*. In this section we outline how this philosophy leads naturally to the concept of a Hopf algebra.

Let X be a finite set. Then

$$A = \mathbb{C}^X = \{f : X \to \mathbb{C} |\ f \text{ function}\}$$

is a unital algebra over $\mathbb{C}$ of finite dimension. Indeed, A is an algebra over a field $\Bbbk$ if

(*i*) A is a vector space over $\Bbbk$,

(*ii*) A has a multiplication

$$m : A \times A \to A, \quad (f, g) \mapsto fg, \text{ with } (fg)(x) = f(x)g(x),$$

which is associative, *i.e.* $(fg)h = f(gh)$ for all $f, g, h \in A$.

(*iii*) A has a unit

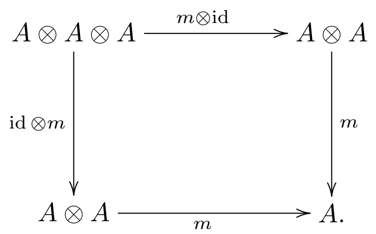
$$u : \Bbbk \to A, \quad \lambda \mapsto \lambda \cdot 1_A,$$

where $1_A = u(1_\Bbbk)$ is the unit of the algebra and the image of u is contained in the centre of A.

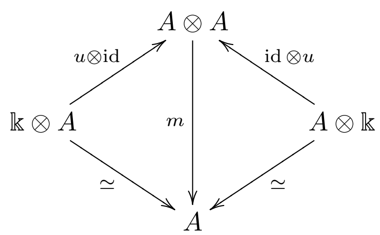
Since m is a bilinear map (*i.e.* linear in each component), we may consider it as a linear map

$$m : A \otimes A \to A, \quad f \otimes g \mapsto fg, \text{ with } (fg)(x) = f(x)g(x).$$

In this case, the associativity can be described by the commutative diagram



The corresponding diagram for the unit is the following



Note that in this case, A is commutative, that is $(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$ for all $f, g \in A, x \in X$. Another way of saying this is the following:

Definition 2.1 Let V and W be two k-vector spaces. The *flip map* τ is the linear map $\tau : V \otimes W \to W \otimes V$ given by $\tau(v \otimes w) = w \otimes v$ for all $v \in V,\ w \in W$.

Then A is commutative if and only if $m \circ \tau = m$ in $A \otimes A$.

Let $x \in X$ and define $\delta_x \in A$ by

$$\delta_x(y) = \delta_{x,y} = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases} \tag{1}$$

Then the set $\{\delta_x\}_{x\in X}$ is a linear basis of A. In particular, $\dim A = |X|$. The multiplication of A is given by $\delta_x\delta_y = \delta_{xy}$ and the unit by $1 = \sum_{x\in X} \delta_x$.

Conversely, given a finite-dimensional commutative algebra over $\mathbb{C}$ without nilpotent elements, we can associate to it the set

$$X = \operatorname{Spec} A = \{\alpha : A \to \mathbb{C} |\ \alpha \text{ is an algebra map}\} = \operatorname{Alg}_{\mathbb{C}}(A, \mathbb{C}),$$

where α is an algebra map if $\alpha(1) = 1$ and $\alpha(ab) = \alpha(a)\alpha(b)$ for all $a, b \in A$. Thus, we have an equivalence

$$\{\text{finite sets}\} \leftrightsquigarrow \{\text{comm. finite-dim. } \mathbb{C}-\text{alg. without nilp. elem.}\}$$

$$X \longmapsto \mathbb{C}^X$$

$$\operatorname{Spec} A \longleftarrow A$$

Clearly, if $A = \mathbb{C}^X$, then $X \subseteq \operatorname{Spec} \mathbb{C}^X$, since for any $x \in X$ we may define $\bar{x}(\alpha) = \alpha(x)$ for all $\alpha \in A$ and it holds that $\bar{x}(\alpha\beta) = (\alpha\beta)(x) = \alpha(x)\beta(x) = \bar{x}(\alpha)\bar{x}(\beta)$ for all $\alpha, \beta \in A$; that is, $\bar{x}$ is an algebra map. The other equality follows from Hilbert's Nullstellensatz.

Suppose now that $X = G$ is a finite group. Then we have maps

$$\begin{aligned} m : G \times G &\to G & u : \{1\} &\to G & S : G &\to G \\ (g, h) &\mapsto gh & 1 &\mapsto e & g &\mapsto g^{-1}, \end{aligned}$$

where m is the product of the group, u gives the unit and S gives the inverse. What are the corresponding maps in $\mathbb{C}^G$? Using these maps we may define in $\mathbb{C}^G$ the following linear maps

$$\begin{aligned} \Delta : \mathbb{C}^G &\to \mathbb{C}^G \otimes \mathbb{C}^G & \varepsilon : \mathbb{C}^G &\to \mathbb{C} & \mathcal{S} : \mathbb{C}^G &\to \mathbb{C}^G \\ f &\mapsto \Delta(f) & f &\mapsto f(1) & f &\mapsto \mathcal{S}(f), \end{aligned}$$

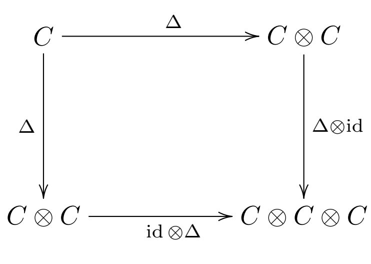
where $\Delta(f)(g \otimes h) := f(gh)$ and $\mathcal{S}(f)(g) = f(g^{-1})$ for all $f \in \mathbb{C}^G, g, h \in G$; that is, Δ, ε, $\mathcal{S}$ are the transpose maps of m, u and S, respectively. In particular, Δ is *coassociative*: since $f((gh)k) = f(g(hk))$ for all $f \in \mathbb{C}^G, g, h, k \in G$, we have that

$$(\Delta \otimes \mathrm{id})\Delta(f)(g \otimes h \otimes k) = f((gh)k) = f(g(hk)) = (\mathrm{id} \otimes\Delta)\Delta(f)(g \otimes h \otimes k),$$

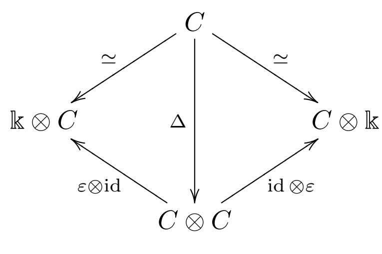
which implies that $(\Delta\otimes\mathrm{id})\Delta(f) = (\mathrm{id} \otimes\Delta)\Delta(f)$ for all $f \in \mathbb{C}^G$. This motivates the following definition. From now on, $\Bbbk$ will denote a field.

Definition 2.2 A $\Bbbk$-*colgebra* is a triple (C, Δ, ε), where C is a $\Bbbk$-vector space $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are linear maps that satisfy the following commutative diagrams

Coassociativity:



Counit:



The map Δ is called *coproduct* or *comultiplication* and the map ε is called *counit*. Usually we refer to a coalgebra only by C, if no confusion arrives. We say that C is *cocommutative* if $\tau \circ \Delta = \Delta$ in C. Note that the commutative diagrams are the same diagrams as in the definition of algebra, but with inverted arrows.

Example 2.3 Let X be a non-empty set and consider the $\Bbbk$-vector space kX with basis $\{e_x\}_{x\in X}$. Then kX is a coalgebra with the linear maps defined on the basis elements by

$$\Delta(e_x) = e_x \otimes e_x, \qquad \varepsilon(e_x) = 1 \quad \text{for all } x \in X.$$

Indeed, since Δ and ε are defined on a basis of kX, it suffices to check the axioms on these elements. For the coassociativity we have

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(e_x) &= (\Delta \otimes \mathrm{id})(e_x \otimes e_x) = \Delta(e_x) \otimes e_x = e_x \otimes e_x \otimes e_x = e_x \otimes \Delta(e_x) = (e_x \otimes e_x) \\ &= (\mathrm{id} \otimes \Delta)(e_x \otimes e_x) = (\mathrm{id} \otimes \Delta)\Delta(e_x),\end{aligned}$$

for all $x \in X$. For the counit we have to check that $e_x = m(\varepsilon \otimes \mathrm{id})\Delta(e_x) = m(\mathrm{id} \otimes \varepsilon)\Delta(e_x)$ for all $x \in X$. But

$$\begin{aligned}m(\varepsilon \otimes \mathrm{id})\Delta(e_x) &= m(\varepsilon \otimes \mathrm{id})(e_x \otimes e_x) = m(\varepsilon(e_x) \otimes e_x) = m(1 \otimes e_x) = e_x \quad \text{and} \\ m(\mathrm{id} \otimes \varepsilon)\Delta(e_x) &= m(\mathrm{id} \otimes \varepsilon)(e_x \otimes e_x) = m(e_x \otimes \varepsilon(e_x)) = m(e_x \otimes 1) = e_x.\end{aligned}$$

Example 2.4 Let G be a finite group. Then $\Bbbk^G$ is a coalgebra with the comultiplication and counit given by

$$\Delta(\delta_g) = \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g} \quad \text{and} \quad \varepsilon(\delta_g) = \delta_{g,e},$$

for all $g \in G$, where $e \in G$ is the identity of the group. Note that Δ and ε are the linear maps induced by the group operations m and u. Indeed,

$$\Delta(\delta_g)(h \otimes t) = \delta_g(ht) = \begin{cases} 1 & \text{if } g = ht \\ 0 & \text{if } g \neq ht \end{cases} = \begin{cases} 1 & \text{if } h^{-1}g = t, \\ 0 & \text{if } h^{-1}g \neq t \end{cases}$$

Since $\{\delta_g\}_{g\in G}$ is a linear basis of $\Bbbk^G$, it follows that $\Delta(\delta_g) = \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g}$. Analogously, $\varepsilon(\delta_g) = \delta_g(e) = \delta_{g,e}$. Since m is associative and u gives the unit, it follows that $\Bbbk^G$ is a coalgebra.

Definition 2.5 *Let C and D be two colgebras with comultiplication Δ_C and Δ_D and counit ε_C and ε_D, respectively.*

(i) A linear map $f : C \to D$ is called a colgebra map if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and $\varepsilon_C = \varepsilon_D \circ f$.

(ii) A linear subspace $E \subseteq C$ is a subcoalgebra if $\Delta(E) \subseteq E \otimes E$.

(iii) A linear subspace $I \subseteq C$ is a coideal if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and $\varepsilon_C(I) = 0$.

The following theorem says that coideals can be viewed as kernels of coalgebra maps and *vice versa*. We leave its proof as exercise.

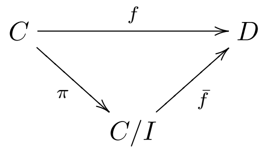
Theorem 2.6 [Sw, Thm. 1.4.7] *Let C be a coalgebra, I a coideal of C and $\pi : C \to C/I$ the canonical linear map onto the quotient vector space. Then*

(a) C/I has a unique coalgebra structure such that π is a coalgebra map. This structure is induced by

$$\widetilde{\Delta}_{C/I} : C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi\otimes\pi} (C/I) \otimes (C/I) \quad \text{and} \quad \varepsilon_{C/I}(c + I) = \varepsilon(C).$$

(b) If $f : C \to D$ is any coalgebra map then Ker f *is a coideal.*

(c) *If $I \subseteq \operatorname{Ker} f$ then there is a unique coalgebra map $\bar{f}$ such that the following diagram commutes*



In particular, from the theorem follows that for all coalgebra C, $C^+ = \operatorname{Ker} \varepsilon \subseteq C$ is a coideal of C, since ε_C is a coalgebra map.

Remark 2.7 In coalgebra theory, one uses usually the Sweedler *sigma notation* for the comultiplication: if $c \in C$, we denote $\Delta(c) = \sum_i a_i \otimes b_i \in C \otimes C$ by

$$\Delta(c) = c_{(1)} \otimes c_{(2)}.$$

For example, the coassociativity axiom of C given by the equality $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, can be written as follows

$$(c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$

for all $c \in C$. We recommend the reader to do the exercises in [Sw, Section 1.2] about sigma notation. The use of this notation turns to be very fruitful in the theory of Hopf algebras.

We have seen in Example 2.4 that the algebra $\Bbbk^G$, G a finite group, has a coalgebra structure. Moreover, these two structures are compatible: with the comultiplication and counit defined above, Δ and ε are algebra maps. Indeed,

$$\Delta(\delta_g \delta_h)(s \otimes t) = \delta_g \delta_h(st) = \delta_g(st)\delta_h(st) = [\Delta(\delta_g)(s \otimes t)][\Delta(\delta_h)(s \otimes t)],$$

for all $s, t \in G$. This implies that $\Delta(\delta_g \delta_h) = \Delta(\delta_g)\Delta(\delta_h)$ for all $g, h \in G$. Moreover, as the unit of $\Bbbk^G$ is given by $1_{\Bbbk^G} = \sum_{g \in G} \delta_g$, we have

$$\Delta(1_{\Bbbk^G}) = \Delta(\sum_{g \in G} \delta_g) = \sum_{g \in G} \Delta(\delta_g) = \sum_{g,h \in G} \delta_h \otimes \delta_{h^{-1}g} = \sum_{k,h \in G} \delta_h \otimes \delta_k = (\sum_{h \in G} \delta_h) \otimes (\sum_{k \in G} \delta_k) = 1_{\Bbbk^G} \otimes 1_{\Bbbk^G}.$$

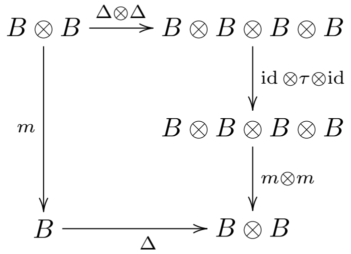
That is, Δ is an algebra map. Analogously, it can be seen that ε is an algebra map and we leave it as exercise for the reader.

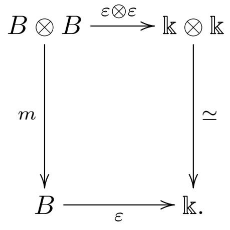
Furthermore, this happens to be equivalent to m and u being algebra maps. This motivates the following definition.

Definition 2.8 A *bialgebra* is a $\Bbbk$-vector space B endowed with an algebra structure (B, m, u) and a coalgebra structure (B, Δ, ε) such that Δ and ε are algebra maps, or equivalently, m and u are coalgebra maps. That is, Δ and ε must satisfy

$$\Delta(ab) = (ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} = \Delta(a)\Delta(b), \qquad \Delta(1) = 1 \otimes 1 \quad \text{and}$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad \varepsilon(1) = 1, \qquad \text{for all } a, b \in B.$$

The corresponding commutative diagrams are the following





As expected, a linear map $f : B \to B'$ between two bialgebras is a *bialgebra map* if f is an algebra map and a coalgebra map. A subspace $I \subseteq B$ is called a *bi-ideal* if it is a two-sided ideal and a coideal. As before, I is a bi-ideal of a bialgebra B if and only if the k-vector space B/I is a bialgebra with the structure induced by the quotient.

Example 2.9 Let G be a finite group. We have seen in Example 2.3 that the linear space $\Bbbk G$ with basis $\{e_g\}_{g\in G}$ is a coalgebra. It also has an algebra structure with unit $1_{\Bbbk G} = e_1$ and the multiplication defined by $e_g e_h = e_{gh}$ for all $g, h \in G$. The algebra $\Bbbk G$ is usually called the *group algebra*. Moreover, since

$$\Delta(e_1) = e_1 \otimes e_1 \quad \text{and} \quad \Delta(e_g e_h) = \Delta(e_{gh}) = e_{gh} \otimes e_{gh} = (e_g \otimes e_g)(e_h \otimes e_h) = \Delta(e_g)\Delta(e_h),$$

it follows that $\Bbbk G$ is a bialgebra.

Example 2.10 Let G be a finite group. In Example 2.4 we showed that $\Bbbk^G$ has a coalgebra structure. It is easy to see that this coalgebra structure is compatible with the algebra structure defined above, giving $\Bbbk^G$ a bialgebra structure. We leave the proof as exercise for the reader.

Example 2.11 Consider the $\Bbbk$-vector space $M_n(\Bbbk)$ of $n \times n$ matrices with coefficients in $\Bbbk$. It has a monoid structure with respect to the multiplication, since not all elements are invertible. Let $\mathcal{O}(M_n(\Bbbk))$ be the commutative algebra over $\Bbbk$ generated by the elements $\{X_{ij} |\ 1 \leq i, j \leq n\}$. As algebra, it is simply the commutative ring of polynomials in n^2 variables

$$\mathcal{O}(M_n(\Bbbk)) = \Bbbk[X_{ij} |\ 1 \leq i, j \leq n].$$

Moreover, $\mathcal{O}(M_n(\Bbbk))$ is a subalgebra of the algebra of functions $\{f : M_n(\Bbbk) \to \Bbbk |\ f \text{ function}\}$ on $M_n(\Bbbk)$, where X_{ij} is the function defined by the matrix coefficient

$$X_{ij}(A) = a_{ij} \qquad \text{for all } A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\Bbbk).$$

If we denote by E_{ij} the matrix with a 1 in the entry (i, j) and zero in all others, the set $\{E_{ij}\}_{1\leq i,j\leq n}$ is a linear basis of $M_n(\Bbbk)$ and the set $\{X_{ij}\}_{1\leq i,j\leq n}$ is the correspondig dual basis with

$$\langle X_{ij}, E_{kl}\rangle = \delta_{ik}\delta_{jl}.$$

Therefore, $\mathcal{O}(M_n(\Bbbk))$ is the algebra of regular functions on $M_n(\Bbbk)$. $\mathcal{O}(M_n(\Bbbk))$ is a bialgebra with the coalgebra structure determined by

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}, \qquad \varepsilon(X_{ij}) = \delta_{ij} \qquad \text{for all } 1 \leq i, j \leq n.$$

Indeed, since $\mathcal{O}(M_n(\Bbbk))$ is generated as a free algebra by the elements $\{X_{ij} |\ 1 \leq i, j \leq n\}$, to define the algebra maps Δ and ε, it suffices to define them on the generators. Moreover, since both maps are uniquely determined by their values on the generators, it is enought to check the coassociativity and the counit axioms on them. For the coassociativity we have

$$(\Delta \otimes \mathrm{id})\Delta(X_{ij}) = (\Delta \otimes \mathrm{id}) \left(\sum_{k=1}^{n} X_{ik} \otimes X_{kj}\right) = \sum_{k=1}^{n} \Delta(X_{ik}) \otimes X_{kj} = \sum_{k,l=1}^{n} X_{il} \otimes X_{lk} \otimes X_{kj} \quad \text{and}$$

$$(\mathrm{id} \otimes\Delta)\Delta(X_{ij}) = (\mathrm{id} \otimes\Delta) \left(\sum_{l=1}^{n} X_{il} \otimes X_{lj}\right) = \sum_{l=1}^{n} X_{il} \otimes \Delta(X_{lj}) = \sum_{k=1}^{n} X_{il} \otimes X_{lk} \otimes X_{kj},$$

for all $1 \leq i, j \leq n$. Thus, Δ is coassociative. For the counit we have

$$\begin{aligned} m(\varepsilon \otimes \mathrm{id})\Delta(X_{ij}) &= m(\varepsilon \otimes \mathrm{id}) \left(\sum_{k=1}^{n} X_{ik} \otimes X_{kj}\right) = m\left(\sum_{k=1}^{n} \varepsilon(X_{ik}) \otimes X_{kj}\right) \\ &= m\left(\sum_{k=1}^{n} \delta_{ik} \otimes X_{kj}\right) = m(1 \otimes X_{ij}) = X_{ij} \qquad \text{and} \\ m(\mathrm{id} \otimes\varepsilon)\Delta(X_{ij}) &= m(\mathrm{id} \otimes\varepsilon) \left(\sum_{k=1}^{n} X_{ik} \otimes X_{kj}\right) = m\left(\sum_{k=1}^{n} X_{ik} \otimes \varepsilon(X_{kj})\right) \\ &= m\left(\sum_{k=1}^{n} X_{ik} \otimes \delta_{kj}\right) = m(X_{ij} \otimes 1) = X_{ij}, \end{aligned}$$

for all $1 \leq i, j \leq n$; which proves that ε is a counit and thus $\mathcal{O}(M_n(\Bbbk))$ is a bialgebra.

Definition 2.12 Let C be a coalgebra and let $c \in C$.

(i) We say that c is a *group-like element* if $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. We denote the set of group-like elements by $G(C)$. If C has a bialgebra structure, then $G(C)$ is a group under the multiplication.

(ii) For $a,\ b \in G(C)$, c is called a (a,b)-*primitive element* if $\Delta(c) = a \otimes c + c \otimes b$. The set of (a,b)-primitive elements is denoted by

$$P_{a,b} = \{c \in C |\ \Delta(c) = a \otimes c + c \otimes b\};$$

in particular, $k(a-b) \subseteq P_{a,b}$. If C is a bialgebra and we take $a = 1 = b$, the elements of $P_{1,1}$ are called simply *primitive elements.*

Examples 2.13 (i) Let G be a finite group and consider the bialgebra structure on $\Bbbk G$. Then $G \subseteq G(\Bbbk G)$, since $\Delta(g) = g \otimes g$ for all $g \in G$. Moreover, on has that $G = G(\Bbbk G)$.

(ii) Consider now the bialgebra structure in $\Bbbk^G$. Then $G(\Bbbk^G) = \mathrm{Alg}_{\Bbbk}(\Bbbk G, \Bbbk) = \widehat{G}$, where $\widehat{G}$ is the character group of G.

Let C be a coalgebra and A an algebra. The set $\mathrm{Hom}_{\Bbbk}(C, A)$ becomes an algebra under the *convolution product* given by

$$(f * g)(c) = f(c_{(1)})g(c_{(2)}) \qquad \text{for all } f, g \in \mathrm{Hom}_{\Bbbk}(C, A),\ c \in C.$$

The unit element in $\mathrm{Hom}_{\Bbbk}(C, A)$ is $u\varepsilon$ with $u\varepsilon(c) = \varepsilon(c)1_A$ for all $c \in C$. Note that when $A = \Bbbk$, then $\mathrm{Hom}_{\Bbbk}(C, \Bbbk) = C^*$ and the algebra structure is the one defined in Exercise 5.

Definition 2.14 Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra and consider $\mathrm{End}_{\Bbbk}(H) = \mathrm{Hom}_{\Bbbk}(H, H)$ as algebra with the convolution product.

(i) An endomorphism $\mathcal{S}$ of H is called an *antipode* for the bialgebra H if

$$\mathcal{S} * \mathrm{id}_H = u\varepsilon = \mathrm{id}_H * \mathcal{S}. \tag{2}$$

That is, $\mathcal{S}$ is the inverse of the identity in $\mathrm{End}_{\Bbbk}(H)$.

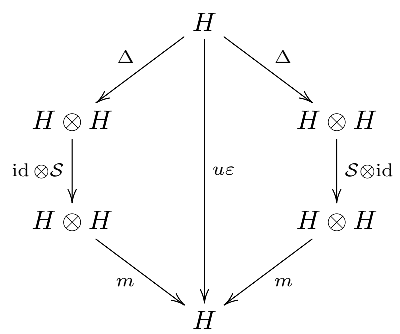
(ii) We say that $(H, m, u, \Delta, \varepsilon, \mathcal{S})$ is a *Hopf algebra* if $\mathcal{S}$ is an antipode for H. We shall denote it only by H if no confusion arrives.

Remarks 2.15 (i) A bialgebra does not need necessarily to have an antipode. If it does, it has only one, simply because $\mathcal{S}$ is the inverse of the identity in $\mathrm{End}_{\Bbbk}(H)$.

(ii) Using the definition of the convolution product, we can re-write equation (2) to the following equality for all $h \in H$

$$\mathcal{S}(h_{(1)})h_{(2)} = u\varepsilon(h) = h_{(1)}\mathcal{S}(h_{(2)}). \tag{3}$$

This equation is usually stated as the antipode axiom for the definition of a Hopf algebra. The corresponding commutative diagram is the following



since by definition we must have that $m(\mathcal{S}\otimes \in)\Delta(h) = m(\mathrm{id} \otimes \mathcal{S})\Delta(h)$, *for all* $h \in H$.

Example 2.16 Let G be a finite group. Then the group algebra $\Bbbk G$ is a Hopf algebra, where the antipode is defined by $\mathcal{S}(e_g) = e_{g^{-1}}$ for all $g \in G$. Indeed, since the elements $\{e_g\}_{g\in G}$ for a linear basis of $\Bbbk G$, $\mathcal{S}$ is defined as a linear map by its values on the basis. Let us check that $\mathcal{S}$ is an antipode by showing equality (3) on e_g for all $g \in G$. Recall that $\Delta(e_g) = e_g \otimes e_g$.

$$\mathcal{S}(e_g)e_g = e_{g^{-1}}e_g = e_{g^{-1}g} = e_1 = 1_{\Bbbk G} = \varepsilon(e_g)1_{\Bbbk G},$$
$$e_g\mathcal{S}(e_g) = e_g e_{g^{-1}} = e_{gg^{-1}} = \varepsilon(e_g)1_{\Bbbk G}.$$

Example 2.17 Let G be a finite group. Then the algebra $\Bbbk^G$ is a Hopf algebra, where the antipode is defined by $\mathcal{S}(\delta_g) = \delta_{g^{-1}}$ for all $g \in G$. Indeed, since the elements $\{\delta_g\}_{g\in G}$ for a linear basis of $\Bbbk^G$, $\mathcal{S}$ is defined as a linear map by its values on the basis. Let us check that $\mathcal{S}$ is an antipode by showing equality (3) on δ_g for all $g \in G$. Recall that $\Delta(\delta_g) = \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g}$.

$$\sum_{h\in G} \mathcal{S}(\delta_h)\delta_{h^{-1}g} = \sum_{h\in G} \delta_{h^{-1}}\delta_{h^{-1}g} == \sum_{h\in G} \delta_{h^{-1},h^{-1}g}\delta_{h^{-1}} = \sum_{h\in G} \delta_{g,1}\delta_{h^{-1}} = \varepsilon(\delta_g)\sum_{h\in G}\delta_{h^{-1}} = \varepsilon(\delta_g)1_{\Bbbk G},$$
$$\sum_{h\in G} \delta_h\mathcal{S}(\delta_{h^{-1}g}) = \sum_{h\in G} \delta_h\delta_{g^{-1}h} == \sum_{h\in G} \delta_{h,g^{-1}h}\delta_h = \sum_{h\in G} \delta_{g^{-1},1}\delta_h = \varepsilon(\delta_{g^{-1}})\sum_{h\in G}\delta_h = \varepsilon(\delta_g)1_{\Bbbk G}.$$

Proposition 2.18 [Sw, Prop. 4.0.1] *Let H be a Hopf algebra with antipode $\mathcal{S}$. Then*

(*a*) $\mathcal{S}(hk) = \mathcal{S}(k)\mathcal{S}(h)$ *and* $\mathcal{S}(1) = 1$, *for all* $h, k \in H$. *In particular,* $\mathcal{S}$ *defines an algebra map* $\mathcal{S} : H \to H^{\rm op}$.

(*b*) $\Delta(\mathcal{S}(h)) = \mathcal{S}(h_{(2)}) \otimes \mathcal{S}(h_{(1)})$ *for all* $h \in H$. *In particular,* $\mathcal{S} : H \to H^{\rm cop}$ *defines a coalgebra map.*

Remark 2.19 Suppose that B is a bialgebra over $\Bbbk$ which is generated as algebra by the elements $\{b_i\}_{i\in I}$. Then to define an antipode on B it is enough to define $\mathcal{S}$ on the generators such that $\mathcal{S} : B \to B^{op}$ is an algebra map and equality (3) holds for all $b_i,\ i \in I$.

As for coalgebras an bialgebras, we have the obvious definitions for Hopf algebra maps and Hopf ideals.

A linear map $f : H \to K$ between two Hopf algebras is called a *Hopf algebra map* if f is a bialgebra map and $f(\mathcal{S}_H(h)) = \mathcal{S}_K(f(h))$ for all $h \in H$. Actually, it can be proved using the uniqueness of the antipode that if $f : H \to K$ is a bialgebra map between two Hopf algebras, then necessarily f preserves the antipode, *i.e.* f is a Hopf algebra map.

A linear subspace I of a Hopf algebra H is called *a Hopf ideal* if I is a bi-ideal and $\mathcal{S}(I) \subseteq I$. Clearly, $I \subseteq H$ is a Hopf ideal if and only if the quotient vector space H/I is a Hopf algebra. For example, $H^+ = \operatorname{Ker}\varepsilon$ is a Hopf ideal of H and it is called the *augmentation ideal* of H.

With this definition, it is straighforward to check that for any finite group G, $(\Bbbk G)^* \simeq \Bbbk^G$ as Hopf algebras. See the exercises.

Example 2.20 Let $(H, m, u, \Delta, \varepsilon, \mathcal{S})$ be a Hopf algebra over $\Bbbk$. Then using the flip map τ, one can easily prove that $(H^{\rm op}, m^{\rm op}, u, \Delta, \varepsilon, \mathcal{S}^{-1})$, $(H^{\rm cop}, m, u, \Delta^{\rm cop}, \varepsilon, \mathcal{S}^{-1})$ and $(H^{\rm op,cop}, m^{\rm op}, u, \Delta^{\rm cop}, \varepsilon, \mathcal{S})$ are Hopf algebras, where $H^{\rm op} = H$ as coalgebra but with the opposite multiplication $m^{\rm op}(h \otimes k) = m \circ \tau(h \otimes k) = m(k \otimes h)$ and $H^{\rm cop} = H$ as algebra but with the opposite comultiplication, that is, $\Delta^{\rm cop}(h) = \tau \circ \Delta(h) = h_{(2)} \otimes h_{(1)}$ for all $h \in H$. We leave the proof of these claims as exercise for the reader.

Example 2.21 Recall from Example 2.11 that for $n = 2$, the algebra

$$\mathcal{O}(M_2(\Bbbk)) = \Bbbk[X_{11}, X_{12}, X_{21}, X_{22}]$$

has a bialgebra structure. To make the notation not so heavy we write from now on $\mathcal{O}(M_2) = \mathcal{O}(M_2(\Bbbk))$ and

$$a = X_{11}, \qquad b = X_{12}, \qquad c = X_{21}, \quad \text{and} \quad d = X_{22}.$$

Then, the comultiplication is determined by

$$\Delta(a) = a\otimes a + b\otimes c, \qquad \Delta(b) = a\otimes b + b\otimes d,$$
$$\Delta(c) = c\otimes a + d\otimes c, \qquad \Delta(d) = c\otimes b + d\otimes d,$$

and the counit by $\varepsilon(a) = 1 = \varepsilon(d)$, $\varepsilon(b) = \varepsilon(c) = 0$.

We define now $\mathcal{O}(SL_2)$ by the commutative algebra generated by the elements a, b, c, d satisfying the relation $ad - bc = 1$. For short we write

$$\mathcal{O}(SL_2) = \Bbbk[a, b, c, d|\ ad - bc = 1].$$

To see that $\mathcal{O}(SL_2)$ inherits the bialgebra structure of $\mathcal{O}(M_2)$, it is enough to prove that the comultiplication Δ and the counits ε are well-defined algebra maps on the quotient, which is the same as saying that the ideal $I = \mathcal{O}(M_2)(ad - bc - 1)$ generated by the element $ad - bc - 1$ is a bi-ideal. Thus we have to prove that $\varepsilon(ad-bc) = \varepsilon(1) = 1$ and $\Delta(ad-bc) = (ad-bc)\otimes(ad-bc) = \Delta(1) = 1\otimes 1$ hold in $\mathcal{O}(M_n)$. Indeed, let $A = \mathcal{O}(M_2)$ and denote by $\pi : A \to A/I$ the canonical quotient. By Theorem 2.6, the comultimplication $\Delta_{A/I}$ is induced by the composition $(\pi\otimes\pi)\Delta$:

$$\begin{array}{ccc} A & \xrightarrow{\ \Delta\ } & A\otimes A \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi\otimes\pi} \\ A/I & & A/I\otimes A/I \end{array}$$

Hence, we have to see that $I \subseteq \operatorname{Ker}(\pi\otimes\pi)\Delta$. But if $t = ad - bc$ and $\Delta(t) = 1$, then

$$\begin{aligned}\Delta(A(t-1)) &= \Delta(A)\Delta(t-1) = (A\otimes A)(t\otimes t - 1\otimes 1) = (A\otimes A)((t-1)\otimes t + 1\otimes(t-1))\\ &= A(t-1)\otimes At + A\otimes A(t-1) \subseteq I\otimes A + A\otimes I,\end{aligned}$$

which implies that $(\pi\otimes\pi)\Delta(I) = 0$. Analogously, $\varepsilon_{A/I}$ is induced by $\varepsilon : A \to \Bbbk$ and $I \subseteq \operatorname{Ker}\varepsilon$ since $\varepsilon(t) = 1$. Thus

$$\begin{aligned}\varepsilon(ad-bc) &= \varepsilon(a)\varepsilon(d) - \varepsilon(b)\varepsilon(c) = 1 \qquad \text{and}\\ \Delta(ad-bc) &= \Delta(a)\Delta(d) - \Delta(b)\Delta(b) = (a\otimes a + b\otimes c)(c\otimes b + d\otimes d) - (a\otimes b + b\otimes d)(c\otimes a + d\otimes c)\\ &= ac\otimes ab + bc\otimes cb + ad\otimes ad + bd\otimes cd - bc\otimes da - bd\otimes dc - ac\otimes ba - ad\otimes bc\\ &= ad\otimes(ad-bc) - bc\otimes(da-cb) = (ad-bc)\otimes(ad-bc).\end{aligned}$$

Thus $\mathcal{O}(SL_2)$ is a bialgebra. This implies in particular that the *determinant* $t = ad-bc$ is a group-like element in $\mathcal{O}(M_n)$.

Furthermore, $\mathcal{O}(SL_2)$ is a Hopf algebra with the antipode given by

$$\mathcal{S}(a) = d, \qquad \mathcal{S}(b) = -b, \qquad \mathcal{S}(c) = -c, \qquad \text{and} \qquad \mathcal{S}(d) = a.$$

If we write

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d\\ c\otimes a + d\otimes c & c\otimes b + d\otimes d\end{pmatrix},$$

we can write

$$\begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b)\\ \mathcal{S}(c) & \mathcal{S}(d)\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

Note that the *antipode matrix* is given by the inverse matrix, since the determinant is equal to 1 in $\mathcal{O}(SL_2)$. Actually, $\mathcal{O}(M_n)$ is not a Hopf algebra with the bialgebra structure defined above, since the determinant t is a group-like element which is not invertible. Notably, adding an inverse t^{-1} to $\mathcal{O}(M_n)$ is enough to give a Hopf algebra structure on the localization $\mathcal{O}(M_n)[t^{-1}]$ of $\mathcal{O}(M_n)$ at t^{-1}. This Hopf algebra is called $\mathcal{O}(GL_n)$ and corresponds to the algebra of regular functions on $GL_n(\Bbbk)$. Another way to obtain a Hopf algebra is to take the quotient by the relation $t = 1$, which defines $\mathcal{O}(SL_n)$.

In matrix notation, the coassociativity follows from the equality

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right),$$

and the counit from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To prove that $\mathcal{S}$ defines an antipode for $\mathcal{O}(SL_2)$, by Remark 2.19 we have to prove first that $\mathcal{S} : \mathcal{O}(SL_2) \to \mathcal{O}(SL_2)^{\mathrm{op}}$ is a well-defined algebra map, and then check equation (3) for the generators. Since $\mathcal{S}(1) = 1$ and $\mathcal{S}(ad - bc) = \mathcal{S}(ad) - \mathcal{S}(bc) = \mathcal{S}(d)\mathcal{S}(a) - \mathcal{S}(c)\mathcal{S}(b) = ad - (-c)(-b) = ad - cb = ad - bc$, it follows that $\mathcal{S} : \mathcal{O}(SL_2) \to \mathcal{O}(SL_2)^{\mathrm{op}}$ is a well-defined algebra map.

To check equation (3) for the generators is equivalent to prove the following matrix equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix} = \begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which follows from the equality $ad - bc = 1$ in $\mathcal{O}(SL_2)$ and we leave it as exercise.

Remark 2.22 Recall that $SL_2(\Bbbk)$ is the subgroup of matrices of $GL_2(\Bbbk)$ given by

$$SL_2(\Bbbk) = \{A \in \Bbbk^{2\times 2} : \det A = 1\}.$$

Then, $\mathcal{O}(SL_2(\Bbbk))$ is the commutative algebra of rational functions on $SL_2(\Bbbk)$ generated by the matrix coefficients via

$$a(A) = a_{11}, \qquad b(A) = a_{12}, \qquad c(A) = a_{21} \quad \text{and} \quad d(A) = a_{22},$$

for all $A = (a_{ij})_{1\leq i,j\leq 2}$. Note that

$$(ad - bc)(A) = a(A)d(A) - b(A)c(A) = a_{11}a_{22} - a_{12}a_{21} = \det A = 1.$$

Moreover, every matrix $A \in SL_2(\Bbbk)$ defines an element of the group $\mathrm{Alg}_{\Bbbk}(\mathcal{O}(SL_2(\Bbbk)), \Bbbk)$ of algebra maps from $\mathcal{O}(SL_2(\Bbbk))$ to $\Bbbk$, by

$$A(a) = a_{11}, \qquad A(b) = a_{12}, \qquad A(c) = a_{21}, \qquad A(d) = a_{22} \quad \text{and} \quad A(1) = 1.$$

It is well-defined since $A(ad - bc) = A(a)A(d) - A(b)A(c) = a_{11}a_{22} - a_{12}a_{21} = \det A = 1$. Conversely, every element α of $\mathrm{Alg}_{\Bbbk}(\mathcal{O}(SL_2(\Bbbk)), \Bbbk)$ defines a matrix in $SL_2(\Bbbk)$ by

$$\begin{pmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{pmatrix},$$

and it holds that $\alpha(a)\alpha(d) - \alpha(b)\alpha(c) = \alpha(ad - bc) = 1$. Hence we have a group isomorphism

$$SL_2(\Bbbk) \simeq \mathrm{Alg}_{\Bbbk}(\mathcal{O}(SL_2(\Bbbk)), \Bbbk).$$

We have constructed Hopf algebras coming from groups, which are commutative and represent algebras of functions on these groups. We end this section with the following theorem that states that if the field $\Bbbk$ is algebraically closed of characteristic zero, then all commutative Hopf algebras arise in this way.

Theorem 2.23 [Cartier] *Let $\Bbbk$ be an algebraically closed field of characteristic zero.*

(*a*) *Let H be a finite-dimensional commutative Hopf algebra. Then H is isomorphic to $\Bbbk^G$, where G is the finite group given by $G = \mathrm{Spec}(H) = \mathrm{Alg}_{\Bbbk}(H, \Bbbk)$.*

(*b*) *Let H be a commutative Hopf algebra, then H is isomorphic to the algebra of regular functions $\mathcal{O}(G)$ on a (pro) algebraic group G.*

2.1 Exercises

1) Prove that the set $\{\delta_x\}_{x\in X}$ defined in (1) is a linear basis of $A = \Bbbk^X$ and it is an algebra with the multiplication given by $\delta_x\delta_y = \delta_{xy}$ for all $x, y \in X$ and the unit by $1 = \sum_{x\in X} \delta_x$.
2) Let G be a finite group. Prove that $\Bbbk^G$ is a coalgebra whose dimension is equal to the order of the group.
3) Let C be a $\Bbbk$-vector space with basis $\{c_m |\ m \in \mathbb{N} \cup \{0\}\}$. Prove that C is a coalgebra with comultiplication Δ and counit ε defined for all $m \in \mathbb{N} \cup \{0\}$ by

$$\Delta(c_m) = \sum_{i=0}^{m} c_i \otimes c_{m-i}, \qquad \varepsilon(c_m) = \delta_{0,m}.$$

4) Let C be a $\Bbbk$-vector space with basis $\{s, c\}$. Prove that C is a coalgebra with comultiplication Δ and counit ε defined by

$$\begin{aligned} \Delta(s) &= s \otimes c + c \otimes s, \qquad & \varepsilon(s) = 0,\\ \Delta(c) &= c \otimes c - s \otimes s, \qquad & \varepsilon(s) = 1. \end{aligned}$$

5) Let C be coalgebra over $\Bbbk$.

(a) Prove that the dual space $C^* = \{f : C \to \Bbbk |\ f \text{ is linear}\}$ is an algebra with the multiplication and unit defined by

$$(f \cdot g)(c) = f(c_{(1)})g(c_{(2)}) \text{ and } 1(c) = \varepsilon(c) \quad \text{for all } f, g \in C^*, c \in C,$$

where $\Delta(c) = c_{(1)} \otimes c_{(2)}$ is the comultiplication of $c \in C$.

(b) Prove that D is a subcoalgebra of C if and only if $D^\perp = \{f : C \to \Bbbk |\ f(D) = 0\}$ is a two-sided ideal of C^*.

(c) Prove that I is a coideal of C if and only if $I^\perp = \{f : C \to \Bbbk |\ f(I) = 0\}$ is a subalgebra of C^*.

6) Prove Theorem 2.6.
7) Let A be a finite-dimensional associative unital $\Bbbk$-algebra.

(a) Prove that A^* is a coalgebra. Hint: Use that $(A \otimes A)^* \simeq A^* \otimes A^*$.

(b) Prove that B is a subalgebra of A if and only if $B^\perp = \{f : A \to \Bbbk |\ f(B) = 0\}$ is a coideal of A^*.

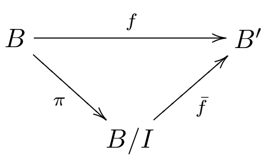
(c) Prove that I is a two-sided ideal of A if and only if $I^\perp = \{f : A \to \Bbbk |\ f(I) = 0\}$ is a subcoalgebra of A^*.

8) Let B be a $\Bbbk$-vector space endowed with an algebra structure (B, m, u) and a coalgebra structure (B, Δ, ε). Prove that Δ and ε are algebra maps if and only if m and u are coalgebra maps.
9) Let B be a bialgebra, I a bi-ideal of B and $\pi : B \to B/I$ the canonical linear map onto the quotient vector space. Then

(a) B/I has a unique bialgebra structure such that π is a bialgebra map.

(b) If $f : B \to B'$ is any bialgebra map then $\operatorname{Ker} f$ is a bi-ideal.

(c) If $I \subseteq \operatorname{Ker} f$ then there is a unique bialgebra map $\bar{f}$ such that the following diagram commutes



10) Let G be a finite group and consider the bialgebra structure on $\Bbbk G$ defined above. Prove that $G = G(\Bbbk G)$.

11) Let G be a finite group and consider the bialgebra structure on $\Bbbk^G$ defined above. Prove that $G(\Bbbk^G) = \mathrm{Alg}_{\Bbbk}(\Bbbk G, \Bbbk) = \widehat{G}$, where $\widehat{G}$ is the character group of G.

12) Let G be a finite group. Prove that $(\Bbbk G)^* \simeq \Bbbk^G$ as Hopf algebras. <u>Hint</u>: Use that $\Bbbk^G \subseteq (\Bbbk G)^*$ via $\langle \delta_g, e_h \rangle = \delta_{gh}$ for all $g, h \in G$.

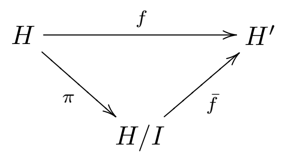
13) Let H be a finite-dimensional Hopf algebra over $\Bbbk$. Prove that H^* is a Hopf algebra.

14) Let H be a Hopf algebra, I a Hopf ideal of H and $\pi : H \to H/I$ the canonical linear map onto the quotient vector space. Then

(*a*) H/I has a unique Hopf algebra structure such that π is a Hopf algebra map.

(*b*) If $f : H \to H'$ is any Hopf algebra map then $\operatorname{Ker} f$ is a Hopf ideal.

(*c*) If $I \subseteq \operatorname{Ker} f$ then there is a unique Hopf algebra map $\bar{f}$ such that the following diagram commutes



15) Let $(H, m, u, \Delta, \varepsilon, \mathcal{S})$ be a Hopf algebra over a field $\Bbbk$. Prove that $(H^{\mathrm{op}}, m^{\mathrm{op}}, u, \Delta, \varepsilon, \mathcal{S}^{-1})$, $(H^{\mathrm{cop}}, m, u, \Delta^{\mathrm{cop}}, \varepsilon, \mathcal{S}^{-1})$ and $(H^{\mathrm{op},\Delta}, m^{\mathrm{op}}, u, \Delta^{\Delta}, \varepsilon, \mathcal{S})$ are Hopf algebras.

16) Prove Proposition 2.18.

17) Prove that the group homomorphism

$$\mathrm{Alg}_{\Bbbk}(\mathcal{O}(SL_2(\Bbbk)), \Bbbk) \xrightarrow{\varphi} SL_2(\Bbbk),$$
$$\alpha \mapsto \begin{pmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{pmatrix}$$

is an isomorphism with inverse ψ determined by

$$\psi(A)(a) = a_{11}, \qquad \psi(A)(b) = a_{12}, \qquad \psi(A)(c) = a_{21} \quad \text{and} \quad \psi(A)(d) = a_{22},$$

for all $A = (a_{ij})_{1 \le i,j \le 2}$.

18) Let A be a $\Bbbk$-algebra. The *finite dual* or *Sweedler dual* of A is given by

$$A^{\circ} = \{f \in A^* |\ f(I) = 0, \text{ for some two-sided ideal } I \text{ of } A \text{ such that } \dim A/I < \infty\}.$$

Let $(A, m, u, \Delta, \varepsilon, \mathcal{S})$ be a Hopf algebra. Prove that A° is a Hopf algebra with the structural maps given by

$$\begin{array}{rlrcl}
m_{A^\circ} = & \Delta^* : A^\circ \otimes A^\circ \to A^\circ & \Delta^*(f \otimes g)(a) & = & (f \otimes g)\Delta(a), \\
u_{A^\circ} = & \varepsilon^* : \Bbbk \to A^\circ & \varepsilon^*(\lambda)(a) & = & \lambda\varepsilon(a), \\
\Delta_{A^\circ} = & m^* : A^\circ \to A^\circ \otimes A^\circ & m^*(f)(a \otimes b) & = & f(ab), \\
\varepsilon_{A^\circ} = & u^* : A^\circ \to \Bbbk & u^*(f) & = & f(1), \\
\mathcal{S}_{A^\circ} = & \mathcal{S}^* : A^\circ \to A^\circ & \mathcal{S}^*(f)(a) & = & f(\mathcal{S}(a)),
\end{array}$$

for all $a,\ b \in A,\ f,\ g \in A^{\circ}$. In particular, if A is finite-dimensional, then $A^{\circ} = A^*$ and whence $(A^*, \Delta^*, \varepsilon^*, m^*, u^*, \mathcal{S}^*)$ is a Hopf algebra.

3 Quantum groups

From now on we will assume that $\Bbbk$ is an algebraically closed field of characteristic zero. In the last section we saw that to to any commutative Hopf algebra corresponds a group, and conversely, to any group corresponds a commutative Hopf algebra

$$\text{Groups} \qquad G \rightsquigarrow \mathcal{O}(G) \qquad \text{Commutative Hopf algebras}$$

$$\mathrm{Alg}_{\Bbbk}(A, \Bbbk) \leftsquigarrow A$$

and this bijection is an equivalence

$$\mathrm{Alg}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C}) = G \leftrightsquigarrow \mathcal{O}(G)$$

Grothendieck's philosophy was extended to quantum groups by Drinfel'd, who stated that one should *quantize* classical coordinate rings such as $\mathcal{O}(G)$ by deforming them to non-commutative Hopf algebras, and that one should study new Hopf algebras as if they consisted of *non-commuting functions* on a non-existing object, namely a *quantum group* corresponding to G.

$$G_q \qquad \leftrightsquigarrow \qquad \mathcal{O}_q(G) \qquad \text{noncommutative Hopf algebras}$$

Thus, quantum groups do not exist as objects, only their algebras of functions. As a convention, the function algebras themselves are called quantum groups.

There is no rigorous, universally accepted definition of the term quantum group. However, it is generally agreed that this term includes certain deformation of classical objects associated to algebraic groups or to semisimimple Lie algebras. To date no axiomatic definition of this family of algebras has been given, nor a complete formulation of properties an algebra should satisfy in order to qualify as a quantum analogue of a given classical coordinate ring. Thus, it is a field driven much more by examples than by axioms.

Some authors define quantum groups as non-commutative and non-cocommutative Hopf algebras. In this notes, we will follow Drinfeld's convention: *the category of quantum groups is the opposite category of Hopf algebras*. That is, as objects quantum groups are Hopf algebras, but the morphisms are the opposite ones. This is because of the following: if Γ is a subgroup of G, then there is a Hopf algebra surjection $\mathcal{O}(G) \twoheadrightarrow \mathcal{O}(\Gamma)$ between the algebras of functions on them.

$$\Gamma \hookrightarrow G \leftrightsquigarrow \mathcal{O}(G) \twoheadrightarrow \mathcal{O}(\Gamma)$$

$$\Gamma_q \hookrightarrow G_q \leftrightsquigarrow \mathcal{O}_q(G) \twoheadrightarrow \mathcal{O}_q(\Gamma)$$

We define in this chapter the first easiest examples of quantum groups that illustrate the theory, $\mathcal{O}_q(SL_2(\Bbbk))$ and $U_q(\mathfrak{sl}_2)(\Bbbk)$.

3.1 Quantum SL_2

Let $q \in \Bbbk^\times = \Bbbk \smallsetminus \{0\}$.

Definition 3.1 The algebra $\mathcal{O}_q(M_2(\Bbbk))$ is the algebra generated by the elements a, b, c, d satisfying the relations

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd, \quad bc = cb, \quad ad - da = (q^{-1} - q)bc.$$

Clearly, when $q = 1$ we have that $\mathcal{O}_1(M_2(\Bbbk)) = \mathcal{O}(M_2(\Bbbk))$, and if $q \neq 1$, then $\mathcal{O}_q(M_2(\Bbbk))$ is not commutative.

Theorem 3.2 (a) *There exist algebra maps*

$$\Delta : \mathcal{O}_q(M_2(\Bbbk)) \to \mathcal{O}_q(M_2(\Bbbk)) \otimes \mathcal{O}_q(M_2(\Bbbk)), \qquad \varepsilon : \mathcal{O}_q(M_2(\Bbbk)) \to \Bbbk,$$

uniquely determined by

$$\Delta(a) = a\otimes a + b\otimes c, \qquad \Delta(b) = a\otimes b + b\otimes d,$$
$$\Delta(c) = c\otimes a + d\otimes c, \qquad \Delta(d) = c\otimes b + d\otimes d,$$
$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0$$

(*b*) *With these morphisms, the algebra $\mathcal{O}_q(M_2(\Bbbk))$ is a bialgebra which is neither commutative nor cocommutative if $q \neq 1$.*

(*c*) *If $\det_q := ad - q^{-1}bc = da - qbc$, then $\Delta(\det_q) = \det_q \otimes \det_q$ and $\varepsilon(\det_q) = 1$, that is, $\det_q$ is a group-like element in $\mathcal{O}_q(M_2(\Bbbk))$. Moreover, it is central.*

Proof. (*a*) In order to prove that Δ and ε are well-defined algebra maps, it is enough to show that the relations hold under Δ and ε, *e.g.* $\Delta(ba) = q\Delta(ab)$.

$$\begin{aligned}
\Delta(ba) &= \Delta(b)\Delta(a) = (a\otimes b + b\otimes d)(a\otimes a + b\otimes c)\\
&= a^2\otimes ba + ab\otimes bc + ba\otimes da + b^2\otimes dc,\\
q\Delta(ab) &= q(a\otimes a + b\otimes c)(a\otimes b + b\otimes d) = qa^2\otimes ab + qab\otimes ad + qba\otimes cb + qb^2\otimes cd\\
&= a^2\otimes qab + ba\otimes (da + (q^{-1}-q)bc) + qba\otimes bc + b^2\otimes qcd\\
&= a^2\otimes ba + ba\otimes da + q^{-1}ba\otimes bc - qba\otimes bc + qba\otimes bc + b^2\otimes dc\\
&= a^2\otimes ba + ba\otimes da + ab\otimes bc + b^2\otimes dc.
\end{aligned}$$

Analogously, one can prove that $\Delta(db) = q\Delta(bd)$, $\Delta(ca) = q\Delta(ac)$, $\Delta(dc) = q\Delta(cd)$, $\Delta(bc) = \Delta(cb)$ and $\Delta(ad - da) = (q^{-1} - q)\Delta(bc)$, and we leave it as exercise for the reader. For ε it is completely analogous. Indeed,

$$\begin{aligned}
\varepsilon(ba) &= \varepsilon(b)\varepsilon(a) = 0 = q\varepsilon(ab) = q\varepsilon(a)\varepsilon(b)\\
\varepsilon(db) &= \varepsilon(d)\varepsilon(b) = 0 = q\varepsilon(bd) = q\varepsilon(b)\varepsilon(d)\\
\varepsilon(bc) &= \varepsilon(b)\varepsilon(c) = 0 = \varepsilon(cb) = \varepsilon(c)\varepsilon(b)\\
\varepsilon(dc) &= \varepsilon(d)\varepsilon(c) = 0 = q\varepsilon(cd) = q\varepsilon(c)\varepsilon(d)\\
\varepsilon(ca) &= \varepsilon(c)\varepsilon(a) = 0 = q\varepsilon(ac) = q\varepsilon(a)\varepsilon(c)\\
\varepsilon(ad-da) &= \varepsilon(a)\varepsilon(d) - \varepsilon(d)\varepsilon(a) = 0 = (q^{-1}-q)\varepsilon(bc) = (q^{-1}-q)\varepsilon(b)\varepsilon(c).
\end{aligned}$$

(*b*) Since the coalgebra structure defined on $\mathcal{O}_q(M_2(\Bbbk))$ is the same as the one defined on $\mathcal{O}(M_n(\Bbbk))$, it follows that $\mathcal{O}_q(M_2(\Bbbk))$ is a coalgebra, that is, ε is a counit and Δ is coassociative. Since both maps are algebra maps, it follows that $\mathcal{O}_q(M_2(\Bbbk))$ is indeed a bialgebra. Clearly, it is not commutative if $q \neq 1$, and it is not cocommtuative since $\Delta(a) = a\otimes a + b\otimes c \neq a\otimes a + c\otimes b = \tau\circ\Delta(a)$.
(*c*) Let $\det_q = ad - q^{-1}bc$. Then

$$\begin{aligned}
\Delta(\det_q) &= \Delta(a)\Delta(d) - q^{-1}\Delta(b)\Delta(c)\\
&= (a\otimes a + b\otimes c)(c\otimes b + d\otimes d) - q^{-1}(a\otimes b + b\otimes d)(c\otimes a + d\otimes c)\\
&= ac\otimes ab + ad\otimes ad + bc\otimes cb + bd\otimes cd - q^{-1}bc\otimes da - q^{-1}bd\otimes dc\\
&\qquad - q^{-1}ac\otimes ba - q^{-1}ad\otimes bc\\
&= ac\otimes ab + ad\otimes (ad - q^{-1}bc) + bc\otimes cb + bd\otimes cd - q^{-1}bc\otimes da - bd\otimes q^{-1}dc - ac\otimes q^{-1}ba\\
&= ac\otimes ab + ad\otimes (ad - q^{-1}bc) + bc\otimes cb + bd\otimes cd - q^{-1}bc\otimes da - bd\otimes cd - ac\otimes ab\\
&= ad\otimes (ad - q^{-1}bc) + bc\otimes cb - q^{-1}bc\otimes da\\
&= ad\otimes (ad - q^{-1}bc) + bc\otimes cb - q^{-1}bc\otimes (ad - (q^{-1}-q)bc)\\
&= ad\otimes (ad - q^{-1}bc) + bc\otimes cb - q^{-1}bc\otimes ad + q^{-2}bc\otimes bc - bc\otimes bc\\
&= ad\otimes (ad - q^{-1}bc) - q^{-1}bc\otimes (ad - q^{-1}bc)\\
&= (ad - q^{-1}bc)\otimes (ad - q^{-1}bc) = \det_q\otimes\det_q.
\end{aligned}$$

Clearly, $\varepsilon(\det_q) = \varepsilon(a)\varepsilon(d) - q^{-1}\varepsilon(b)\varepsilon(c) = 1$. Thus, $\det_q$ is a group-like element. To see that it is central, it is enough to verify it on the generators:

$$\begin{aligned}
\det_q a &= (ad - q^{-1}bc)a = ada - q^{-1}bca = a(ad - (q^{-1} - q)bc) - q^{-1}q^2abc \\
&= a(ad - q^{-1}bc) + qabc - qabc = a\det_q, \\
\det_q b &= (ad - q^{-1}bc)b = adb - q^{-1}bcb = q^{-1}qbad - bbc \\
&= b(ad - q^{-1}bc) = b\det_q, \\
\det_q c &= (ad - q^{-1}bc)c = adc - q^{-1}bcc = q^{-1}qbad - cbc \\
&= c(ad - q^{-1}bc) = c\det_q, \\
\det_q d &= (ad - q^{-1}bc)d = add - q^{-1}bcd = (da + (q^{-1} - q)bc)d - q^{-1}bcd \\
&= dad + q^{-1}bcd - qbcd - q^{-1}bcd = dad - qq^{-2}dbc = d\det_q.
\end{aligned}$$

□

Definition 3.3 [Mn] We define $\mathcal{O}_q(SL_2(\Bbbk))$ as the $\Bbbk$-algebra given by the quotient

$$\mathcal{O}_q(SL_2(\Bbbk)) = \mathcal{O}_q(M_2(\Bbbk))/(\det_q - 1),$$

where $(\det_q - 1)$ is the two-sided ideal of $\mathcal{O}_q(M_2(\Bbbk))$ generated by the element $\det_q - 1$.

In other words, the algebra $\mathcal{O}_q(SL_2(\Bbbk))$ can be presented as the $\Bbbk$-algebra generated by the elements a, b, c, d satisfying the relations

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd, \quad bc = cb, \quad ad - da = (q^{-1} - q)bc, \quad ad - q^{-1}bc = 1.$$

Clearly, when $q = 1$ we have that $\mathcal{O}_1(SL_2(\Bbbk)) = \mathcal{O}(SL_2(\Bbbk))$, and if $q \neq 1$, then $\mathcal{O}_q(SL_2(\Bbbk))$ is not commutative.

Since $\det_q$ is a central group-like element, the ideal $(\det_q - 1)$ of $\mathcal{O}_q(M_2(\Bbbk))$ is indeed a bi-ideal and thus $\mathcal{O}_q(SL_2(\Bbbk))$ is a bialgebra with the comultiplication and counit definesd on the generators as in $\mathcal{O}_q(M_2(\Bbbk))$.

Theorem 3.4 $\mathcal{O}_q(SL_2(\Bbbk))$ is a Hopf algebra with the antipode determined by

$$\begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix},$$

that is, $\mathcal{S}(a) = d$, $\mathcal{S}(b) = -qb$, $\mathcal{S}(c) = -q^{-1}c$ and $\mathcal{S}(d) = a$.

Proof. First we have to prove that $\mathcal{S} : \mathcal{O}_q(SL_2(\Bbbk)) \to \mathcal{O}_q(SL_2(\Bbbk))^{\text{op}}$ is a well-defined algebra map:

$$\begin{aligned}
\mathcal{S}(ba) &= \mathcal{S}(a)\mathcal{S}(b) = d(-qb) = -qdb = -q^2bd = q\mathcal{S}(b)\mathcal{S}(a) = q\mathcal{S}(ab), \\
\mathcal{S}(db) &= \mathcal{S}(b)\mathcal{S}(d) = (-qb)a = -q^2ab = q\mathcal{S}(d)\mathcal{S}(b) = q\mathcal{S}(bd), \\
\mathcal{S}(ca) &= \mathcal{S}(a)\mathcal{S}(c) = d(-q^{-1}c) = -q^{-1}dc = -cd = q\mathcal{S}(c)\mathcal{S}(a) = q\mathcal{S}(ac), \\
\mathcal{S}(dc) &= \mathcal{S}(c)\mathcal{S}(d) = (-q^{-1}c)a = -ac = q\mathcal{S}(d)\mathcal{S}(c) = q\mathcal{S}(cd), \\
\mathcal{S}(bc) &= \mathcal{S}(c)\mathcal{S}(b) = (-q^{-1}c)(-qb) = cb = bc\mathcal{S}(b)\mathcal{S}(c) = \mathcal{S}(cb), \\
\mathcal{S}(ad - da) &= \mathcal{S}(ad) - \mathcal{S}(da) = \mathcal{S}(d)\mathcal{S}(a) - \mathcal{S}(a)\mathcal{S}(d) = ad - da = (q^{-1} - q)bc = (q^{-1} - q)cb \\
&= (q^{-1} - q)\mathcal{S}(c)\mathcal{S}(b) = (q^{-1} - q)\mathcal{S}(bc), \\
\mathcal{S}(ad - q^{-1}bc) &= \mathcal{S}(ad) - q^{-1}\mathcal{S}(bc) = \mathcal{S}(d)\mathcal{S}(a) - q^{-1}\mathcal{S}(c)\mathcal{S}(b) = ad - q^{-1}q^{-1}qcb \\
&= ad - q^{-1}cb = ad - q^{-1}bc = 1 = \mathcal{S}(1).
\end{aligned}$$

To prove that $\mathcal{S}$ defines an antipode for $\mathcal{O}_q(SL_2(\Bbbk))$, we have to check equation (3) for the generators. As for the case of $\mathcal{O}(SL_2(\Bbbk))$, this is equivalent to prove the following matrix equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix} = \begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which follows from the defining relations of $\mathcal{O}_q(SL_2(\Bbbk))$ and we leave it as exercise for the reader.

□

Remark 3.5 The quantum group $\mathcal{O}_q(SL_2(\Bbbk))$ corresponds to the *quantized* coordinate ring of $SL_2(\Bbbk)$ generated by the matrix coefficients.

3.2 Quantum $\mathfrak{sl}_2$

There is another group associated to $SL_2(\Bbbk)$. In effect, $SL_2(\mathbb{C})$ is not only a group but also a smooth manifold, *i.e.* a *Lie group*. As such, it has a tangent space at the identity, which is a *Lie algebra* and it is called $\mathfrak{sl}_2$. Moreover, one can see that

$$\mathfrak{sl}_2 = \{A \in M_2(\mathbb{C}) : \ \mathrm{tr}(A) = 0\}.$$

The quantum group we introduce in this section corresponds to the deformation in one parameter of the enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$. The deformation uses the classification of semisimple Lie algebras over an algebraically closed field of characteristic zero, done by Cartan and Killing. Thus, the field $\Bbbk$ is an arbitrary field with these properties.

The origins of the subject of quantum groups lie in mathematical physics, where the term quantum comes from. The starting point of the study of this subject lies in the Quantum Inverse Scattering Method, with the aim of solving certain integrable quantum systems. A key ingredient in this method is the *Quantum Yang-Baxter Equation* (QYBE). While there is no general method for solving the QYBE, it was discovered in the early 1980s that some solutions could be constructed from the representation theory of certain algebras resembling deformations of enveloping algebras of semisimple Lie algebras. The first such deformation of $U(\mathfrak{sl}_2)$, arose from a paper of Kulish and Reshetikhin [KR]. In the mid-1980s, Drinfeld and Jimbo independently discovered analogous deformations corresponding to arbitrary semisimple Lie algebras [Dr, Dr2, Ji].

Definition 3.6 Let $q \in \Bbbk^\times$, $q \neq \pm 1$. We define $U_q(\mathfrak{sl}_2)(\Bbbk)$ as the $\Bbbk$-algebra generated by the elements E, F, K, K^{-1} satisfying the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

If no confusion arrives, we denote this algebra simply by $U_q(\mathfrak{sl}_2)$. Observe that it is non-commutative. Moreover, it has the following properties.

Proposition 3.7

(*a*) $U_q(\mathfrak{sl}_2)(\Bbbk)$ is a noetherian domain with no zero divisors.

(*b*) The set $\{E^i F^j K^l : \ i, j \in \mathbb{N}_0, l \in \mathbb{Z}\}$ is a linear basis of $U_q(\mathfrak{sl}_2)(\Bbbk)$. In particular, $U_q(\mathfrak{sl}_2)(\Bbbk)$ is infinite-dimensional.

Proof. See [K, Prop. VI.1.4] or [BG, Chp. 1.3].

□

Remark 3.8 The basis given by part (*b*) is called a PBW-basis of $U_q(\mathfrak{sl}_2)(\Bbbk)$.

Theorem 3.9

(*a*) There exist algebra maps

$$\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2), \qquad \varepsilon : U_q(\mathfrak{sl}_2) \to \Bbbk,$$

uniquely determined by

$$\begin{aligned}
\Delta(K) &= K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},\\
\Delta(E) &= 1 \otimes E + E \otimes K,\\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1,\\
\varepsilon(K) &= \varepsilon(K^{-1}) = 1, \qquad \varepsilon(E) = \varepsilon(F) = 0.
\end{aligned}$$

(*b*) With these morphisms, $U_q(\mathfrak{sl}_2)$ is a bialgebra which is non-commutative and non-cocommutative. In particular, the powers of K are group-like elements, $E \in P_{K,1}$ and $F \in P_{1,K^{-1}}$.

(*c*) $U_q(\mathfrak{sl}_2)$ is a Hopf algebra with the antipode determined by

$$\mathcal{S}(E) = -EK^{-1}, \qquad \mathcal{S}(F) = -KF, \qquad \mathcal{S}(K) = K^{-1} \quad \text{and} \quad \mathcal{S}(K^{-1}) = K.$$

Proof. (*a*) We first show that Δ defines an algebra map. For this it is enough to check that the ideal of relations is a coideal, or equivalently, that the following equalities hold

$$\Delta(KK^{-1}) = \Delta(K^{-1}K) = 1 \otimes 1 = \Delta(1), \qquad \Delta(KEK^{-1}) = q^2\Delta(E),$$
$$\Delta(KFK^{-1}) = q^{-2}\Delta(F), \qquad \Delta(EF - FE) = \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right).$$

The first relations are clear since

$$\Delta(KK^{-1}) = \Delta(K)\Delta(K^{-1}) = (K \otimes K)(K^{-1} \otimes K^{-1}) = KK^{-1} \otimes KK^{-1} = 1 \otimes 1.$$

For the others we have

$$\begin{aligned}
\Delta(KEK^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1})\\
&= (K \otimes KE + KE \otimes K^2)(K^{-1} \otimes K^{-1})\\
&= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K\\
&= 1 \otimes q^2E + q^2E \otimes K = q^2\Delta(E).
\end{aligned}$$

The relation for F is completely analogous and we leave it as exercise for the reader. For the last relation we have

$$\begin{aligned}
\Delta(EF - FE) &= \Delta(E)\Delta(F) - \Delta(F)\Delta(E)\\
&= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K)\\
&= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K - K^{-1} \otimes FE - K^{-1}E \otimes FE\\
&\quad - F \otimes E - FE \otimes K\\
&= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K + EK^{-1} \otimes KF - K^{-1}E \otimes FK\\
&= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K + q^2q^{-2}K^{-1}E \otimes FK - K^{-1}E \otimes FK\\
&= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K\\
&= K^{-1} \otimes \left(\frac{K - K^{-1}}{q - q^{-1}}\right) + \left(\frac{K - K^{-1}}{q - q^{-1}}\right) \otimes K\\
&= \frac{1}{q - q^{-1}}(K^{-1} \otimes K - K^{-1} \otimes K^{-1} + K \otimes K - K^{-1} \otimes K)\\
&= \frac{1}{q - q^{-1}}(K \otimes K - K^{-1} \otimes K^{-1})\\
&= \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right).
\end{aligned}$$

Now we check that ε is a well-defined algebra map by showing that the equalities in the relations hold after applying ε:

$$\begin{aligned}
\varepsilon(KK^{-1}) &= \varepsilon(K)\varepsilon(K^{-1}) = 1.1 = \varepsilon(1) = \varepsilon(K^{-1})\varepsilon(K) = \varepsilon(K^{-1}K)\\
\varepsilon(KEK^{-1}) &= \varepsilon(K)\varepsilon(E)\varepsilon(K^{-1}) = 1.0.1 = 0 = q^2\varepsilon(E)\\
\varepsilon(KFK^{-1}) &= \varepsilon(K)\varepsilon(F)\varepsilon(K^{-1}) = 1.0.1 = 0 = q^{-2}\varepsilon(F)\\
\varepsilon(EF-FE) &= \varepsilon(E)\varepsilon(F) - \varepsilon(F)\varepsilon(E) = 0 = \varepsilon\left(\frac{K-K^{-1}}{q-q^{-1}}\right) = \frac{\varepsilon(K)-\varepsilon(K^{-1})}{q-q^{-1}}.
\end{aligned}$$

(b) To prove that $U_q(\mathfrak{sl}_2)$ is a bialgebra, we need to show that $(U_q(\mathfrak{sl}_2), \Delta, \varepsilon)$ is a coalgebra, since by (a), we know that Δ and ε are algebra maps. We prove that ε is a counit and Δ is coassociative by checking the equalities

$$m(\varepsilon\otimes \mathrm{id})\Delta = m(\mathrm{id}\otimes\varepsilon)\Delta = \mathrm{id} \quad \text{and} \quad (\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta,$$

on the generators. We begin by the counit:

$$\begin{aligned}
m(\varepsilon\otimes \mathrm{id})\Delta(K) &= m(\varepsilon\otimes \mathrm{id})(K\otimes K) = m(\varepsilon(K)\otimes K) = m(1\otimes K) = K \quad \text{and}\\
m(\mathrm{id}\otimes\varepsilon)\Delta(K) &= m(\mathrm{id}\otimes\varepsilon)(K\otimes K) = m(K\otimes\varepsilon(K)) = m(K\otimes 1) = K,\\
m(\varepsilon\otimes \mathrm{id})\Delta(K^{-1}) &= m(\varepsilon\otimes \mathrm{id})(K^{-1}\otimes K^{-1}) = m(\varepsilon(K^{-1})\otimes K^{-1}) = m(1\otimes K^{-1}) = K^{-1} \quad \text{and}\\
m(\mathrm{id}\otimes\varepsilon)\Delta(K^{-1}) &= m(\mathrm{id}\otimes\varepsilon)(K^{-1}\otimes K^{-1}) = m(K^{-1}\otimes\varepsilon(K^{-1})) = m(K^{-1}\otimes 1) = K^{-1},\\
m(\varepsilon\otimes \mathrm{id})\Delta(E) &= m(\varepsilon\otimes \mathrm{id})(1\otimes E + E\otimes K) = m(\varepsilon(1)\otimes E + \varepsilon(E)\otimes K) = m(1\otimes E) = E \text{ and}\\
m(\mathrm{id}\otimes\varepsilon)\Delta(E) &= m(\mathrm{id}\otimes\varepsilon)(1\otimes E + E\otimes K) = m(1\otimes\varepsilon(E) + E\otimes\varepsilon(K)) = m(E\otimes 1) = E,\\
m(\varepsilon\otimes \mathrm{id})\Delta(F) &= m(\varepsilon\otimes \mathrm{id})(K^{-1}\otimes F + F\otimes 1) = m(\varepsilon(K^{-1})\otimes F + \varepsilon(F)\otimes 1) = m(1\otimes F) = F\\
m(\mathrm{id}\otimes\varepsilon)\Delta(F) &= m(\mathrm{id}\otimes\varepsilon)(K^{-1}\otimes F + F\otimes 1) = m(K^{-1}\otimes\varepsilon(F) + F\otimes\varepsilon(1)) = m(F\otimes 1) = F.
\end{aligned}$$

For the coassociativity we have

$$\begin{aligned}
(\Delta\otimes \mathrm{id})\Delta(K) &= (\Delta\otimes \mathrm{id})(K\otimes K) = \Delta(K)\otimes K = K\otimes K\otimes K \quad \text{and}\\
(\mathrm{id}\otimes\Delta)\Delta(K) &= (\mathrm{id}\otimes\Delta)(K\otimes K) = K\otimes\Delta(K) = K\otimes K\otimes K,\\
(\Delta\otimes \mathrm{id})\Delta(K^{-1}) &= (\Delta\otimes \mathrm{id})(K^{-1}\otimes K^{-1}) = \Delta(K^{-1})\otimes K^{-1} = K^{-1}\otimes K^{-1}\otimes K^{-1} \quad \text{and}\\
(\mathrm{id}\otimes\Delta)\Delta(K^{-1}) &= (\mathrm{id}\otimes\Delta)(K^{-1}\otimes K^{-1}) = K^{-1}\otimes\Delta(K^{-1}) = K^{-1}\otimes K^{-1}\otimes K^{-1},\\
(\Delta\otimes \mathrm{id})\Delta(E) &= (\Delta\otimes \mathrm{id})(1\otimes E + E\otimes K) = \Delta(1)\otimes E + \Delta(E)\otimes K =\\
&= 1\otimes 1\otimes E + 1\otimes E\otimes K + E\otimes K\otimes K \quad \text{and}\\
(\mathrm{id}\otimes\Delta)\Delta(E) &= (\mathrm{id}\otimes\Delta)(1\otimes E + E\otimes K) = 1\otimes\Delta(E) + E\otimes\Delta(K)\\
&= 1\otimes 1\otimes E + 1\otimes E\otimes K + E\otimes K\otimes K,\\
(\Delta\otimes \mathrm{id})\Delta(F) &= (\Delta\otimes \mathrm{id})(K^{-1}\otimes F + F\otimes 1) = \Delta(K^{-1})\otimes F + \Delta(F)\otimes 1 =\\
&= K^{-1}\otimes K^{-1}\otimes F + K^{-1}\otimes F\otimes 1 + F\otimes 1\otimes 1 \quad \text{and}\\
(\mathrm{id}\otimes\Delta)\Delta(F) &= (\mathrm{id}\otimes\Delta)(K^{-1}\otimes F + F\otimes 1) = K^{-1}\otimes\Delta(F) + F\otimes\Delta(1)\\
&= K^{-1}\otimes K^{-1}\otimes F + K^{-1}\otimes F\otimes 1 + F\otimes 1\otimes 1.
\end{aligned}$$

Thus Δ is coassociative and clearly $U_q(\mathfrak{sl}_2)$ is not cocommutative since $\tau\circ\Delta \neq \Delta$ because

$$\Delta(E) = 1\otimes E + E\otimes K \neq E\otimes 1 + K\otimes E = \tau\circ\Delta(E).$$

(c) To prove that $\mathcal{S}$ is an antipode, we have to check first that $\mathcal{S} : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\mathrm{op}}$ is an algebra map and then that equality (3) holds for all generators of $U_q(\mathfrak{sl}_2)$. To show that $\mathcal{S}$ defines an algebra map, we have to verify that the equalities of the relations hold when appying $\mathcal{S}$, but using the opposite multiplication, for example $\mathcal{S}(KEK^{-1}) = \mathcal{S}(K^{-1})\mathcal{S}(E)\mathcal{S}(K) = q^2\mathcal{S}(E)$, but

$$\mathcal{S}(KEK^{-1}) = \mathcal{S}(K^{-1})\mathcal{S}(E)\mathcal{S}(K) = K(-EK^{-1})K^{-1} = -KEK^{-1}K^{-1} = -q^2EK^{-1} = q^2\mathcal{S}(E).$$

Clearly it holds for K and K^{-1} and the computation for F is completely analogous to the computation above and we leave it as exercise. For the last relation we have

$$\begin{aligned}\mathcal{S}(EF-FE) &= \mathcal{S}(F)\mathcal{S}(E)-\mathcal{S}(E)\mathcal{S}(F) = (-KF)(-EK^{-1})-(-EK^{-1})(-KF)\\ &= KFEK^{-1}-EF = KFq^2K^{-1}E-EF = q^{-2}q^2KK^{-1}FE-EF = FE-EF\\ &= -\frac{K-K^{-1}}{q-q^{-1}} = \frac{\mathcal{S}(K)-\mathcal{S}(K^{-1})}{q-q^{-1}} = \mathcal{S}\left(\frac{K-K^{-1}}{q-q^{-1}}\right).\end{aligned}$$

Thus, $\mathcal{S}: U_q(\mathfrak{sl}_2)\to U_q(\mathfrak{sl}_2)^{\mathrm{op}}$ is a well-defined algebra map. Now we prove that the equality $m(\mathrm{id}\otimes\mathcal{S})\Delta = u\varepsilon = m(\mathcal{S}\otimes\mathrm{id})\Delta$ holds by verifying it on the generators:

$$\begin{aligned}m(\mathrm{id}\otimes\mathcal{S})\Delta(K) &= m(\mathrm{id}\otimes\mathcal{S})(K\otimes K) = m(K\otimes\mathcal{S}(K)) = m(K\otimes K^{-1}) = 1 \quad\text{and}\\ m(\mathcal{S}\otimes\mathrm{id})\Delta(K) &= m(\mathcal{S}\otimes\mathrm{id})(K\otimes K) = m(\mathcal{S}(K)\otimes K) = m(K^{-1}\otimes K) = 1,\\ m(\mathrm{id}\otimes\mathcal{S})\Delta(F) &= m(\mathrm{id}\otimes\mathcal{S})(K^{-1}\otimes F+F\otimes 1) = m(K^{-1}\otimes\mathcal{S}(F)+F\otimes\mathcal{S}(1))\\ &= m(K^{-1}\otimes(-KF)+F\otimes 1) = K^{-1}(-KF)+F = 0 \quad\text{and}\\ m(\mathcal{S}\otimes\mathrm{id})\Delta(F) &= m(\mathcal{S}\otimes\mathrm{id})(K^{-1}\otimes F+F\otimes 1) = m(\mathcal{S}(K^{-1})\otimes F+\mathcal{S}(F)\otimes 1)\\ &= m(K\otimes F+(-KF)\otimes 1) = KF-KF = 0.\end{aligned}$$

The equalities for K^{-1} and E are again completely analogous and we leave it as exercise.

□

3.2.1 Quantum Borel subgroups

The quantum groups $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ we introduce here are actually quantum quotients, since they are constructed as Hopf subalgebras of $U_q(\mathfrak{sl}_2)$. Their terminology comes from classical Lie theory, since $U(\mathfrak{b}^+)\subseteq U(\mathfrak{sl})$.

These quantum groups are just the subalgebras of $U_q(\mathfrak{sl}_2)$ generated by a subset of the generators, $K^{\pm 1}, E$ in the first case, and $K^{\pm 1}, F$ in the second. In particular, they can be described as algebras as follows

$$U_q(\mathfrak{b}^+) = \Bbbk\{K, K^{-1}, E:\ KK^{-1} = 1 = K^{-1}K, KEK^{-1} = q^2E\}$$

$$U_q(\mathfrak{b}^-) = \Bbbk\{K, K^{-1}, F:\ KK^{-1} = 1 = K^{-1}K, KFK^{-1} = q^{-2}F\}.$$

They are called the *Quantum Borel subgroups* of $U_q(\mathfrak{sl}_2)$ and the positive and the negative part of $U_q(\mathfrak{sl}_2)$, respectively. They are also usually denoted by $U_q(\mathfrak{sl}_2)^+$ and $U_q(\mathfrak{sl}_2)^-$, or $U_q(\mathfrak{sl}_2)^{\geq 0}$ and $U_q(\mathfrak{sl}_2)^{\leq 0}$, respectively.

From Theorem 3.9 it follows that both algebras are indeed Hopf subalgebras of $U_q(\mathfrak{sl}_2)$, since $\Delta(E) = 1\otimes E+E\otimes K\in U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{b}^+)$ and $\Delta(F) = K^{-1}\otimes F+F\otimes 1\in U_q(\mathfrak{b}^-)\otimes U_q(\mathfrak{b}^-)$. Clearly, the multiplication on $U_q(\mathfrak{sl}_2)$ induces a surjective Hopf algebra map

$$U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{b}^-)\twoheadrightarrow U_q(\mathfrak{sl}_2).$$

Thus, the quantum group may be considered in some sense as a *double* object. This notion was formally defined by Drinfeld who showed a way of producing solutions of the QYBE from Hopf algebras which are *Drinfeld doubles*.

3.2.2 Relation between $\mathcal{O}_q(SL_2)$ and $U_q(\mathfrak{sl}_2)$

Up to now, we have introduced the quantum groups $\mathcal{O}_q(SL_2)$ and $U_q(\mathfrak{sl}_2)$. In fact, these objects are deeply related and in some sense, they are dual of each other. In this section we describe in a formal way this relation, which holds for quantum groups associated to arbitrary semisimple Lie algebras.

Definition 3.10 [Tk2] Given two bialgebras U and H, a *Hopf pairing* between them is a bilinear form

$$\langle -,-\rangle : H\times U \to \Bbbk,$$

such that for all $u,v\in U$, $h,k\in H$, it holds

$$\begin{aligned}\langle h,uv\rangle &= \langle h_{(1)},u\rangle\langle h_{(2)},v\rangle,\\ \langle hk,u\rangle &= \langle h,u_{(1)}\rangle\langle k,u_{(2)}\rangle,\\ \langle 1,u\rangle &= \varepsilon(u), \qquad \langle h,1\rangle = \varepsilon(h).\end{aligned}$$

If moreover U and H are Hopf algebras, from the equalities above it follows that $\langle \mathcal{S}(h),u\rangle = \langle h,\mathcal{S}(u)\rangle$, for all $u\in U$, $h\in H$.

Let φ and ψ be the linear maps from U to H^* and from H to U^* given by

$$\begin{aligned}\varphi : U\to H^*, &\qquad \varphi(u)(h) = \langle h,u\rangle \qquad \text{for all} \quad u\in U, h\in H,\\ \psi : H\to U^*, &\qquad \psi(u)(h) = \langle h,u\rangle, \qquad \text{for all} \quad h\in H, u\in U.\end{aligned}$$

If φ and ψ are injective, we say that the Hopf pairing is *perfect.* The following theorem states the duality between the quantum groups defined above.

Theorem 3.11 [K, Thm. VII.4.4], [BG, I.9.23]. There is a Hopf pairing between $\mathcal{O}_q(SL_2(\Bbbk))$ and $U_q(\mathfrak{sl}_2)(\Bbbk)$ given by

$$\begin{array}{llll}\langle a,K\rangle = q, & \langle a,K^{-1}\rangle = q^{-1}, & \langle a,E\rangle = 0, & \langle a,F\rangle = 0,\\ \langle b,K\rangle = 0, & \langle b,K^{-1}\rangle = 0, & \langle b,E\rangle = 1, & \langle b,F\rangle = 0,\\ \langle c,K\rangle = 0, & \langle c,K^{-1}\rangle = 0, & \langle c,E\rangle = 0, & \langle c,F\rangle = 1,\\ \langle a,K\rangle = q, & \langle a,K^{-1}\rangle = q^{-1}, & \langle a,E\rangle = 0, & \langle a,F\rangle = 0,\\ \langle d,K\rangle = q^{-1}, & \langle d,K^{-1}\rangle = q, & \langle d,E\rangle = 0, & \langle d,F\rangle = 0.\end{array}$$

If q is not a root of unity, the Hopf pairing is perfect. In particular $\mathcal{O}_q(SL_2(\Bbbk)) \hookrightarrow U_q(\mathfrak{sl}_2)(\Bbbk)^*$ and $U_q(\mathfrak{sl}_2)(\Bbbk) \hookrightarrow \mathcal{O}_q(SL_2(\Bbbk))^*$.

Remark 3.12 It holds in fact that the image of $\psi : \mathcal{O}_q(SL_2(\Bbbk)) \to U_q(\mathfrak{sl}_2)(\Bbbk)^*$ is the *finite dual* $U_q(\mathfrak{sl}_2)(\Bbbk)^\circ$ of $U_q(\mathfrak{sl}_2)(\Bbbk)$, that is $\mathcal{O}_q(SL_2(\Bbbk)) \simeq U_q(\mathfrak{sl}_2)(\Bbbk)^\circ$ as Hopf algebras. This statement holds in general for simply connected semisiple Lie groups G with corresponding semisimple Lie algebras $\mathfrak{g}$, *i.e.* $\mathcal{O}_q(G)(\Bbbk) \simeq U_q(\mathfrak{g})(\Bbbk)^\circ$. However, it is not true that $\mathcal{O}_q(G)(\Bbbk)^\circ \simeq U_q(\mathfrak{g})(\Bbbk)$. See [Tk] for a detailled discussion on the case $G = SL_2$.

3.3 Small quantum groups

From now on we assume that q is a primitive root of unity of order $\ell \neq 1$, $\ell > 2$. In this section we introduce the Frobenius-Lusztig kernels $\mathbf{u}_q(\mathfrak{sl}_2)$, which are finite-dimensional Hopf algebras. They are constructed as quotients of $U_q(\mathfrak{sl}_2)$ by the two-sided ideal generated by some central elements. For this reason they are called *small quantum groups.* We begin first by describing these central elements.

Lemma 3.13 The elements E^ℓ, F^ℓ, K^ℓ and $K^{-\ell}$ are central in $U_q(\mathfrak{sl}_2)$.

Proof. It is a straightforward computation and we leave it as exercise. For example, K^ℓ is central since it commutes with all the generators of $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned}K^\ell E &= q^{2\ell}EK^\ell = EK^\ell\\ K^\ell F &= q^{-2\ell}FK^\ell = FK^\ell.\end{aligned}$$

For E^ℓ the computations are similar. Indeed, $KE^\ell = q^{2\ell}E^\ell K = E^\ell K$ and

$$\begin{aligned}
E^\ell F &= E^{\ell-1}\left(FE + \frac{K-K^{-1}}{q-q^{-1}}\right) = E^{\ell-1}FE + \frac{1}{q-q^{-1}}E^{\ell-1}(K-K^{-1}) \\
&= E^{\ell-2}\left(FE + \frac{K-K^{-1}}{q-q^{-1}}\right)E + \frac{1}{q-q^{-1}}E^{\ell-1}(K-K^{-1}) \\
&= E^{\ell-2}FE^2 + \frac{q^2}{q-q^{-1}}E^{\ell-1}K - \frac{q^{-2}}{q-q^{-1}}E^{\ell-1}K^{-1} + \frac{1}{q-q^{-1}}E^{\ell-1}(K-K^{-1}) \\
&= E^{\ell-2}FE^2 + \frac{q^2+1}{q-q^{-1}}E^{\ell-1}K - \frac{q^{-2}+1}{q-q^{-1}}E^{\ell-1}K^{-1} \\
&\ \vdots \\
&= FE^\ell + \frac{q^{2\ell}+q^{2(\ell-1)}+q^2+1}{q-q^{-1}}E^{\ell-1}K - \frac{q^{-2\ell}+q^{-2(\ell-1)}+q^{-2}+1}{q-q^{-1}}E^{\ell-1}K^{-1} \\
&= FE^\ell,
\end{aligned}$$

since $q^{-2\ell}+q^{-2(\ell-1)}+q^{-2}+1 = 0 = q^{2\ell}+q^{2(\ell-1)}+q^2+1$ because q is an ℓ-th root of unity.

□

Since the elements $K^\ell - 1, E^\ell, F^\ell$ are central in $U_q(\mathfrak{sl}_2)$, the ideal I generated by these elements is a two-sided ideal. Moreover, from the quantum binomial formula it follows

$$\begin{aligned}
\Delta(K^\ell - 1) &= (K^\ell - 1)\otimes K^\ell + 1\otimes(K^\ell - 1), \\
\Delta(E^\ell) &= 1\otimes E^\ell + E^\ell \otimes K^\ell, \\
\Delta(F^\ell) &= K^{-\ell}\otimes F^\ell + F^\ell \otimes 1,
\end{aligned}$$

see Exercises 10), 11) and 12). Hence, $\Delta(I) \subseteq I\otimes U_q(\mathfrak{sl}_2) + U_q(\mathfrak{sl}_2)\otimes I$. Since $\varepsilon(K^\ell - 1) = 0 = \varepsilon(E^\ell) = \varepsilon(F^\ell)$, it follows that I is a bi-ideal. Furthermore, it is a Hopf ideal since

$$\begin{aligned}
\mathcal{S}(K^\ell - 1) &= K^{-\ell} - 1 = K^{-\ell}(1-K^\ell) \in I \\
\mathcal{S}(E^\ell) &= \mathcal{S}(E)^\ell = (-EK^{-1})^\ell = (-1)^\ell q^{-2(1+2+\cdots+\ell-1)}E^\ell K^{-\ell} = (-1)^\ell q^{-\ell(\ell-1)}E^\ell K^{-\ell} \\
&= (-1)^\ell E^\ell K^{-\ell} \in I \\
\mathcal{S}(F^\ell) &= \mathcal{S}(F)^\ell = (-KF)^\ell = q^{2(1+2+\cdots+\ell-1)}K^\ell F^\ell = q^{\ell(\ell-1)}K^\ell F^\ell \\
&= K^\ell F^\ell \in I.
\end{aligned}$$

The *Frobenius-Lusztig Kernel* is defined as the Hopf algebra given by the quotient

$$\mathbf{u}_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)/I.$$

It is also called the restricted enveloping algebra of $\mathfrak{sl}_2$. It can be also presented as the algebra generated by the elements $\{K, E, F\}$ satisfying the relations

$$K^\ell = 1, \qquad E^\ell = 0 = F^\ell, \qquad KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \quad \text{and} \quad EF - FE = \frac{K-K^{-1}}{q-q^{-1}}.$$

Lemma 3.14 $\dim \mathbf{u}_q(\mathfrak{sl}_2) = \ell^3$.

Proof. (Sketch) As the set $\{E^iF^jK^m : \ i,j\in\mathbb{N}, m\in\mathbb{Z}\}$ is a linear basis of $U_q(\mathfrak{sl}_2)$, it follows that the set $\{E^iF^jK^m : \ 0 \le i,j,m < \ell\}$ is a linear basis for $\mathbf{u}_q(\mathfrak{sl}_2)$. It is clear that they generate $\mathbf{u}_q(\mathfrak{sl}_2)$ as a linear space. To see that they are linearly independent one may look at the representation theory or use the Diamond Lemma, see for example [AS2].

□

Remarks 3.15 *(i)* By definition, the comultiplication in $\mathbf{u}_q(\mathfrak{sl}_2)$ is determined by

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = 1 \otimes E + E \otimes K \quad \text{and} \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1.$$

In particular, we have that $\mathcal{S}(u) = KuK^{-1}$ for all $u \in \mathbf{u}_q(\mathfrak{sl}_2)$.

(ii) If q is a root of unity, the Hopf algebra $\mathcal{O}_q(SL_2)$ contains a central Hopf subalgebra B isomorphic to $\mathcal{O}(SL_2)$, which is given by

$$B = \Bbbk[a^\ell, b^\ell, c^\ell, d^\ell : \ a^\ell d^\ell - b^\ell c^\ell = 1] \subseteq \mathcal{O}_q(SL_2).$$

Moreover, one has the central exact sequence of Hopf algebras

$$\Bbbk \to \mathcal{O}(SL_2) \to \mathcal{O}_q(SL_2) \to \mathbf{u}_q(\mathfrak{sl}_2)^* \to \Bbbk.$$

There is another small quantum group, which is related to the quantum Borel subgroup $U_q(\mathfrak{b}^+)$ of $U_q(\mathfrak{sl}_2)$. Just take the ideal J of $U_q(\mathfrak{b}^+)$ generated by the central elements $K^\ell - 1$ and E^ℓ. Again, this ideal J is a Hopf ideal and one has the finite-dimensional Hopf algebra

$$\mathbf{u}_q(\mathfrak{b}^+) = U_q(\mathfrak{b}^+)/J,$$

which can be presented as an algebra as follows

$$\mathbf{u}_q(\mathfrak{b}^+) = \Bbbk\{K, E : \ K^\ell = 1, E^\ell = 0, KEK^{-1} = q^2 E\}.$$

The Hopf algebra structure is determined by

$$\begin{array}{cc} \Delta(K) = K \otimes K, & \Delta(E) = 1 \otimes E + E \otimes K, \\ \varepsilon(K) = 1, & \varepsilon(E) = 0 \\ \mathcal{S}(K) = K^{-1}, & \mathcal{S}(E) = -xg. \end{array}$$

Lemma 3.16

(*a*) $\dim \mathbf{u}_q(\mathfrak{b}^+) = \ell^2$.

(*b*) There is an injective Hopf algebra map $\mathbf{u}_q(\mathfrak{b}^+) \hookrightarrow \mathbf{u}_q(\mathfrak{sl}_2)$.

Proof. (*b*) is clear and (*a*) follows from the fact that $\{E^i K^m : \ 0 \leq i, m < \ell\}$ is a basis of $\mathbf{u}_q(\mathfrak{b}^+)$.

□

Remark 3.17 These type of Hopf algebras where defined by Sweedler in the case $\ell = 2$ and Taft in the case $\ell > 2$. They are called now the Sweedler and Taft algebras, respectively, and are presented as follows:

$$T_q = \Bbbk\{g, x : \ g^\ell = 1, x^\ell = 0, gxg^{-1} = qx\}.$$

The Hopf algebra structure is determined by

$$\begin{array}{cc} \Delta(g) = g \otimes g, & \Delta(x) = 1 \otimes x + x \otimes g, \\ \varepsilon(g) = 1, & \varepsilon(x) = 0 \\ \mathcal{S}(g) = g^{-1}, & \mathcal{S}(x) = -xg. \end{array}$$

In particular, $\mathcal{S}^2(h) = ghg^{-1}$ for all $h \in T_q$. With the notation used before we have $\mathbf{u}_q(\mathfrak{b}^+) \simeq T_{q^2}$.

3.4 Exercises

1) Prove that the algebra map $\mathcal{S} : \mathcal{O}_q(SL_2(\Bbbk)) \to \mathcal{O}_q(SL_2(\Bbbk))^{\mathrm{op}}$ defined by

$$\begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

is an antipode for $\mathcal{O}_q(SL_2(\Bbbk))$.
2) Let t be an indeterminate and define the *quantum general linear group* by

$$\mathcal{O}_q(GL_2(\Bbbk)) = \mathcal{O}_q(M_2(\Bbbk))[t]/(t\det_q - 1).$$

Prove that $\mathcal{O}_q(GL_2(\Bbbk))$ is a Hopf algebra with the coalgebra structure induced by the quotient and the antipode given by

$$\begin{pmatrix} \mathcal{S}(a) & \mathcal{S}(b) \\ \mathcal{S}(c) & \mathcal{S}(d) \end{pmatrix} = t \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

3) **Coaction on the quantum plane**. The quantum plane is define as the $\Bbbk$-algebra $\Bbbk_q[x,y]$ given by

$$\Bbbk_q[x,y] = \Bbbk\{x,y \mid yx = qxy\}.$$

There exists a linear map

$$\begin{aligned} \rho : &\Bbbk_q[x,y] \to \mathcal{O}_q(SL_2(\Bbbk)) \otimes \Bbbk_q[x,y], \\ &x \mapsto a \otimes x + b \otimes y, \\ &y \mapsto c \otimes x + d \otimes y, \end{aligned}$$

which in matrix notation can be written as

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}.$$

Prove that ρ defines a coaction of $\mathcal{O}_q(SL_2(\Bbbk))$ on $\Bbbk_q[x,y]$, that is, the following equality holds

$$(\mathrm{id} \otimes \rho)\rho = (\Delta \otimes \mathrm{id})\rho.$$

Moreover, ρ is an algebra map which implies that $\Bbbk_q[x,y]$ is a $\mathcal{O}_q(SL_2(\Bbbk))$-comodule algebra.
4) Prove that $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ are Hopf subalgebras of $U_q(\mathfrak{sl}_2)$.
5) Let U^+ be the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by E, U^- the subalgebra generated by F and U_0 the subalgebra generated by K and K^{-1}. Prove that there is a linear isomorphism

$$U^+ \otimes U_0 \otimes U^- \simeq U_q(\mathfrak{sl}_2)$$

This property is called the *triangular decomposition* and resembles the same property in the classical case.
6) Prove that $\mathcal{S}^2(u) = KuK^{-1}$ for all $u \in U_q(\mathfrak{sl}_2)$.
7) **Chevalley involution**. Prove that there is a unique algebra automorphism w of $U_q(\mathfrak{sl}_2)$ determined by

$$w(E) = F, \qquad w(F) = E \quad \text{and} \quad w(K) = K^{-1},$$

which satisfies $w^2 = \mathrm{id}$.
8) **Limit as** $q \to 1$. We could have presented $U_q(\mathfrak{sl}_2)$ as the algebra generated by the elements E, F, K, K^{-1} and L satisfying the relations

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \qquad EF - FE = L, \qquad KK^{-1} = K^{-1}K = 1,$$
$$(q - q^{-1})L = K - K^{-1}.$$

The presentation given by these 8 relations has the advantage that it makes sense when $q = 1$. Let $U(\mathfrak{sl}_2)$ be the algebra given by

$$U(\mathfrak{sl}_2) = \Bbbk\{H, E, F : \ HE - EH = 2E, HF - FH = -2F, EF - FE = H\}$$

Prove that $U(\mathfrak{sl}_2) \simeq U_1(\mathfrak{sl}_2)/(K-1)$. <u>Hint</u>: Prove first that the relations above imply when $q \neq \pm 1$

$$LE - EL = q(EK + K^{-1}E), \qquad LF - FL = -q(KF + FK^{-1})$$

Then prove that $U_1(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)[K]/(K^2 - 1)$ and then $U(\mathfrak{sl}_2) \simeq U_1(\mathfrak{sl}_2)/(K-1)$.

9) Let q be a primitive root of unity of order ℓ, with $\ell > 2$. Prove that the elements E^ℓ, F^ℓ, K^ℓ and $K^{-\ell}$ are central in $U_q(\mathfrak{sl}_2)$.

10) In the polynomial algebra $\mathbb{Z}[\mathbf{q}]$, $\mathbf{q}$ an indeterminate, we consider the $\mathbf{q}$-binomial coecients

$$\binom{n}{i}_{\mathbf{q}} = \frac{(n)_{\mathbf{q}}!}{(n-i)_{\mathbf{q}}!(i)_{\mathbf{q}}!},$$

where $(n)_{\mathbf{q}}! = (n)_{\mathbf{q}}(n-1)_{\mathbf{q}}\cdots(1)_{\mathbf{q}}$ and $(n)_{\mathbf{q}} = 1+\mathbf{q}+\mathbf{q}^2+\cdots\mathbf{q}^{n-1}$ for all $n\in\mathbb{N}$ and $0\le i\le n$.

(a) Prove the identity

$$\mathbf{q}^k\binom{n}{k}_{\mathbf{q}} + \binom{n}{k-1}_{\mathbf{q}} = \binom{n+1}{k}_{\mathbf{q}} \qquad \text{for all } 0\le k\le n. \tag{4}$$

(b) Prove by induction that $\binom{n}{i}_{\mathbf{q}} \in \mathbb{Z}[\mathbf{q}]$, for all $n\in\mathbb{N}$, $0\le i\le n$.

11) **Quantum binomial formula**. If A is an associative algebra over $\Bbbk$ and $q\in\Bbbk$, then $\binom{n}{i}_q$ denotes the specialization of $\binom{n}{i}_{\mathbf{q}}$ at q. Prove that if $x,y\in A$ are two elements that q-commute, *i.e.* $xy=qyx$, then the following formula holds for every $n\in\mathbb{N}$:

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}_q y^i x^{n-i}. \tag{5}$$

12) Let q be a primitive ℓ-th root of unity. Then by definition we have that $\binom{\ell}{i}_q = 0$ for all $0<i<\ell$. Let E,F be the the generators of $U_q(\mathfrak{sl}_2)$. Prove using the quantum binomial formula that

$$\Delta(E^\ell) = 1\otimes E^\ell + E^\ell\otimes K^\ell \qquad \text{and} \qquad \Delta(F^\ell) = K^{-\ell}\otimes F^\ell + F^\ell\otimes 1.$$

13) Let q be a primitive ℓ-th root of unity and consider the Taft algebra T_q defined above.

(a) Prove that $T_q\simeq T_{q'}$ if and only if $q=q'$.

(b) Prove that T_q is self-dual, that is, $T_q\simeq T_q^*$.

4 Quantum groups and the classification problem of finite-dimensional Hopf algebras

In this last section we sketch how quantum groups get into the scene of the classification problem. First we need some definitions.

Definition 4.1 (i) Let H be a Hopf algebra. We say that H is *semisimple* if it is semisimple as algebra; that is, if its Jacobson radical is zero.

(ii) A coalgebra is called *simple* if it does not contain non-trivial subcoalgebras. It is called *cosemisimple* if it is the sum of simple subcoalgebras.

The following theorem is due to several author, see [LR1], [LR2], [LR3], [R, Prop. 2], [OSch1] and [OSch2]. For a complete proof see [Sch].

Theorem 4.2 Let H be a finite-dimensional Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic zero. The following are equivalent (*a*) H is semisimple.

(*b*) H is cosemisimple.

(*c*) $\mathcal{S}^2 = \mathrm{id}_H$.

(*d*) $\mathrm{Tr}\,\mathcal{S}^2 \neq 0$.

□

From now on we assume that all Hopf algebras are finite-dimensional and the field is algebraically closed of characteristic zero.

Examples 4.3 (*i*) Let G be a finite group. Then $\Bbbk G$ and $\Bbbk^G$ are semisimple Hopf algebras. For instance, in $\Bbbk G$ we have that $\mathcal{S}(e_g) = e_{g^{-1}}$ for all $g \in G$. This implies that $\mathcal{S}^2(e_g) = e_g$ for all $g \in G$. Since $\{e_g\}_{g\in G}$ is a linear basis of $\Bbbk G$, it follows that $\mathcal{S}^2 = \mathrm{id}$. One can see that $\Bbbk^G$ is also semisimple by a direct computation or just use that $(\Bbbk G)^* \simeq \Bbbk^G$.

(*ii*) The Taft algebra T_q is not semisimple since $\mathcal{S}^2(x) = gxg^{-1} = qx \neq x$.

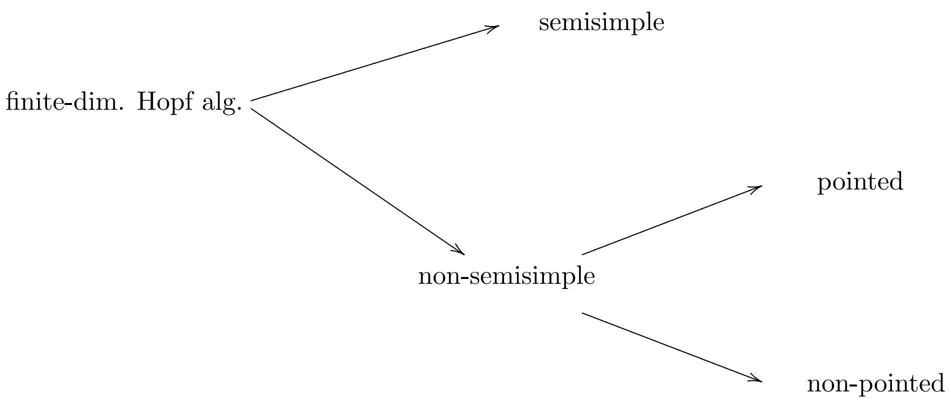
(*iii*) The Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2)$ is not semisimple if $q^2 \neq 2$, since $\mathcal{S}^2(E) = KEK^{-1} = q^2E$.

Definition 4.4 (*i*) The *coradical* of a coalgebra C is the sum of all simple subcoalgebras and it is denoted by C_0. Clearly, C_0 is a cosemisimple subcoalgebra of C.

(*ii*) Let H be a Hopf algebra. We say that H is *pointed* if all simple subcoalgebras are one-dimensional. In particular, this implies that $H_0 = \Bbbk G(H)$.

Example 4.5 The Taft algebras T_q and the Frobenius-Lusztig kernels $\mathbf{u}_q(\mathfrak{sl}_2)$ are pointed.

The study of finite-dimensional Hopf algebras goes through two different directions: the semisimple case and the non-semisimple case.



In the non-semisimple case, a particular subclass was intensively studied: the pointed ones. For example, Andruskiewitsch and Schneider [AS3] proved that all Hopf algebras which are pointed non-semisimple and whose coradical is a finite abelian group G such that $|G|$ is not divisible by a prime number smaller or equal to 7, are variations of Frobenius-Lusztig kernels.

Up to now, the smallest dimension where the classification is unknown is 20.

Let H be a Hopf algebra and let p be a prime number. The classification is known if $\dim H = p, p^2$ and $2p, 2p^2$ with p odd. We describe now the classification for the cases p, p^2 and some known results on dimension p^3.

The classification of the Hopf algebras over $\Bbbk$ of dimension p was obtained by Zhu. It turns out that there are only group algebras of cyclic groups of order p.

Theorem 4.6 [Z] Let H be a Hopf algebra of dimension p. Then H is isomorphic to the group algebra $\Bbbk[\mathbb{Z}/(p)]$.

□

The classification of Hopf algebras of dimension p^2 is due to several authors. The semisimple ones were classified by Masuoka, using that all semisimple Hopf algebras of dimension p^n, $n \in \mathbb{N}$ have a central group-like element. Andruskiewitsch and Schneider prove that the only pointed Hopf algebras of dimension p^2 are the Taft algebras. Finally, Ng completed the classification by showing that a Hopf algebra of dimension p^2 is either semisimple or pointed (non-semisimple).

Theorem 4.7 [Mk1], [AS1], [Ng] Let H be a Hopf algebra of dimension p^2. Then H is isomorphic to one and only one of the following Hopf algebras:

(a) $\Bbbk[\mathbb{Z}/(p^2)]$,

(b) $\Bbbk[\mathbb{Z}/(p) \times \mathbb{Z}/(p)]$

(c) The Taft algebra T_q with $q^p = 1$, $q \neq \pm 1$.

□

We close these notes with the classification of semisimple and pointed non-semisimple Hopf algebras of dimension p^3. Hopf algebras of dimension 8 were classified by Williams [W] in her PhD Thesis. Masuoka [Mk2] and Stefan [St] gave later a different proof of this result. In general, the classification problem of Hopf algebras of dimension p^3 remains open. Nevertheless, the classification is known for the semisimple case and the pointed non-semisimple case. Suppose that p is an odd prime. Semisimple Hopf algebras of dimension p^3 were classified by Masuoka [Mk1]; there are exactly $p + 8$ isomorphism classes:

(a) Three group algebras of abelian groups.

(b) Two group algebras of non-abelian groups and their duals.

(c) $p + 1$ self-dual Hopf algebras which are neither commutative nor cocommutative and they are given by the extensions of $k[\mathbb{Z}/(p) \times \mathbb{Z}/(p)]$ by $k[\mathbb{Z}/(p)]$.

Note that in this case, there are semisimple Hopf algebras which are non-trivial, *i.e.* not isomorphic to a group algebra or the dual of a group algebra.

Non-semisimple pointed Hopf algebras of dimension p^3 were classified by [AS2], [CD] and [SvO], by different methods. It turns out that they consist of variations of small quantum groups. The explicit list is the following, where $q \in \mathbb{G}_p \smallsetminus \{1\}$:

(d) The tensor-product Hopf algebra $T(q) \otimes k[\mathbb{Z}/(p)]$.

(e) $\widetilde{T(q)} := k < g,\ x |\ gxg^{-1} = q^{1/p}x,\ g^{p^2} = 1,\ x^p = 0 >$ ($q^{1/p}$ a p-th root of q), with comultiplication $\Delta(x) = x \otimes g^p + 1 \otimes x,\ \Delta(g) = g \otimes g$.

(f) $\widehat{T(q)} := k < g,\ x|\ gxg^{-1} = qx,\ g^{p^2} = 1,\ x^p = 0 >$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x,\ \Delta(g) = g \otimes g$.

(g) $\mathbf{r}(q) := k < g,\ x|\ gxg^{-1} = qx,\ g^{p^2} = 1,\ x^p = 1 - g^p >$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x,\ \Delta(g) = g \otimes g$.

(h) The Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2) := k < g,\ x,\ y|\ gxg^{-1} = q^2x,\ gyg^{-1} = q^{-2}y,\ g^p = 1,\ x^p = 0,\ y^p = 0,\ xy - yx = g - g^{-1} >$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x,\ \Delta(y) = y \otimes 1 + g^{-1} \otimes y,\ \Delta(g) = g \otimes g$.

(i) The book Hopf algebra $\mathbf{h}(q,m) := k < g,\ x,\ y|\ gxg^{-1} = qx,\ gyg^{-1} = q^m y,\ g^p = 1,\ x^p = 0,\ y^p = 0,\ xy - yx = 0 >$, $m \in \mathbb{Z}/(p) \setminus \{0\}$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes 1 + g^m \otimes y$, $\Delta(g) = g \otimes g$.

Furthermore, there are two examples of non-semisimple but also non-pointed Hopf algebras of dimension p^3, namely

(j) The dual of the Frobenius-Lusztig kernel, $\mathbf{u}_q(\mathfrak{sl}_2)^*$.

(k) The dual of the case (g), $\mathbf{r}(q)^*$.

There are no isomorphisms between different Hopf algebras in the list. Moreover, the Hopf algebras in cases (d), ..., (k) are not isomorphic for different values of $q \in \mathbb{G}_p \setminus \{1\}$, except for the book algebras, where $\mathbf{h}(q,m)$ is isomorphic to $\mathbf{h}(q^{-m^2}, m^{-1})$. In particular, the Hopf algebra $\widetilde{T(q)}$ does not depend, modulo isomorphisms, upon the choice of the p-th root of q.

It is a conjecture that any Hopf algebra H of dimension p^3 is semisimple or pointed or its dual is pointed, that is, H is one of the Hopf algebras of the list (a), ..., (k). In [G] this conjecture is proved under additional assumptions.

4.1 Exercises

1) Let G be a finite group. Prove that the Hopf algebra $\mathbb{k}^G$ is semisimple.

2) Let q be a primitive root of unity. Prove that the Hopf algebras T_q and $\mathbf{u}_q(\mathfrak{sl}_2)$ are pointed and non-semisimple.

3) Prove using the duality between algebras and coalgebras that a finite-dimensional Hopf algebra H is pointed if and only if all simple H-modules are one-dimensional.

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Hopf algebras

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