On collapsing simple racks

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CONICET

New trends in Hopf algebras and tensor categories

June 2-5, 2015, Brussels, Belgium
Joint work with N. Andruskiewitsch and G. Carnovale.


- Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type III. Semisimple classes in $\text{PSL}_n(q)$. Preprint.
Introduction

Background:

We say that a finite group $G$ **collapses** if every finite-dimensional pointed Hopf algebra $H$, with $G(H) \simeq G$ is isomorphic to $\mathbb{C}G$.

- If $G \simeq \mathbb{Z}/p$ is simple abelian, then the classification is known: for $p = 2$ by [N]; for $p > 7$, by [AS3]; for $p = 3, 5, 7$, by [AS1] and [AS4].

- If $G \simeq \mathbb{A}_m$, $m \geq 5$ is alternating, then $G$ collapses [AFGV1].

- If $G$ is a sporadic simple group, then $G$ collapses, except for the groups $G = Fi_{22}, B, M$ [AFGV2], [FV].

- $G = PSL_2(q)$ collapses for $q > 2$ even [FGV2].
Let $G$ be a finite group and $H$ a pointed Hopf algebra with $G(H) \cong G$.

Let $0 = H_{-1} \subset H_0 = \mathbb{C} G(H) \subset H_1 \subset \ldots$ be the coradical filtration of $H$ and $\text{gr} \ H = \bigoplus_{n \in \mathbb{N}_0} H_n/H_{n-1} \cong R \# \mathbb{C} G(H)$.

$R = \bigoplus_{n \in \mathbb{N}_0} R^n$ is a graded Hopf algebra in $\mathbb{C}_G \mathcal{YD}$. Also, the subalgebra of $R$ generated by $V := R^1$ is isomorphic to the Nichols algebra $\mathcal{B}(V)$ of $V$. Hence

$\dim H < \infty \iff \dim R < \infty \implies \dim \mathcal{B}(V) < \infty$.

**Question**

Determine all $V \in \mathbb{C}_G \mathcal{YD}$ with $\dim \mathcal{B}(V) < \infty$.

The following are equivalent [AFGV1]:

- $G$ collapses.
- For every $V \in \mathbb{C}_G \mathcal{YD}$, $\dim \mathcal{B}(V) = \infty$.
- For every *irreducible* $V \in \mathbb{C}_G \mathcal{YD}$, $\dim \mathcal{B}(V) = \infty$. 
For $V \in \mathbb{C}^G \mathcal{YD}$ and $g \in G$ recall

$$V_g = \{v \in V : \delta(v) = g \otimes v\} \subseteq V, \text{ and } V = \bigoplus_{g \in G} V_g,$$

$$\text{supp } V = \{g \in G : V_g \neq \{0\}\} \subseteq G.$$

By the compatibility condition: $\delta(h \cdot v) = hgh^{-1} \otimes h \cdot v$ for all $v \in V_g$ follows that $\text{supp } V$ is the union of conjugacy classes of $G$.

**Fact:**
All irreducible Yetter-Drinfeld modules over $\mathbb{C}^G$ are of the form $M(O, \rho) = \text{Ind}^G_{C_G(g)} V$, $O$ is a conjugacy class of $G$ and $(\rho, V) \in \text{Irr } C_G(g)$ for $g \in O$ fixed. Set $\mathcal{B}(M(O, \rho)) = \mathcal{B}(O, \rho)$.

**Question**
Determine all pairs $(O, \rho)$ with $\dim \mathcal{B}(O, \rho) < \infty$. 

Racks

- $\mathcal{B}(\mathcal{O}, \rho)$ depends only on the underlying braided vector space $(\mathcal{C}\mathcal{O} \otimes V, c^\rho)$.

- Graña [Gr] showed that this class of braided vector spaces in $\mathcal{C}_G \mathcal{YD}$ for $G$ non-abelian can be constructed using racks and rack 2-cocycles.

**Definition**

A **rack** is a non-empty set $X$ endowed with a map $\triangleright : X \times X \to X$ satisfying

(a) $\varphi_x := x \triangleright \_ \text{ is a bijection for any } x \in X$,

(b) $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \text{ for all } x, y, z \in X$.

We denote $\text{Inn}X := \langle \varphi_x, x \in X \rangle < \mathcal{S}_X$. The first example of a rack is a conjugacy class in a group $G$, with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. 
Let $X$ be a finite rack and $n \in \mathbb{N}$. A map $q : X \times X \to \text{GL}_n(\mathbb{C})$ is a 2-cocycle of degree $n$ if

$$q_{x,y \triangleright z}q_{y,z} = q_{x \triangleright y,x \triangleright z}q_{x,z} \quad \text{for all } x, y, z \in X.$$

The cocycle induces a braiding $c^q$ on the vector space $\mathbb{C}X \otimes \mathbb{C}^n$. We denote $\mathcal{B}(X,q)$ its Nichols algebra.

**Question [AFGV1]**

Determine all pairs $(X,q)$, where $X$ is a finite rack and $q$ is a non-principal 2-cocycle, such that $\dim \mathcal{B}(X,q) < \infty$.

**Definition**

A finite rack $X$ *collapses* when $\dim \mathcal{B}(X,q) = \infty$ for every finite faithful 2-cocycle $q$. 
From now all racks are finite.

We say that a rack is:

- **abelian** if \( x \triangleright y = y \) for all \( x, y \in X \).
- **decomposable** if it contains two subracks \( R, S \) such that \( X = R \sqcup S \) and \( R \triangleright S \subseteq S, S \triangleright R \subseteq R \).
- **sober** if every subrack is either abelian or indecomposable.
- **simple** if \( |X| > 1 \) and any rack epimorphism \( X \twoheadrightarrow Y \) is bijective or \( |Y| = 1 \).

**Lemma [AFGV1]**

Let \( X \) be a finite simple rack. Then \( X \) collapses iff for any finite group \( G \) and any \( M \in \mathbb{C}_G \mathcal{YD} \) such that \( X \subseteq \text{supp} \, M \) we have \( \dim \mathcal{B}(M) = \infty \).
Need of criteria to solve the problem without looking at the 2-cocycle.

Andruskiewitsch, Heckenberger and Schneider [AHS] begun the study of $\mathcal{B}(V)$ with $V \in \mathbb{C}_G^G \mathcal{YD}$ semisimple.

Subsequently H & S proved

**Theorem [HS]**

Let $G$ be a finite group, $g, h \in G$ and $V = \bigoplus_{s \in O_g} V_s$, $W = \bigoplus_{t \in O_h} W_t \in \mathbb{C}_G^G \mathcal{YD}$ irreducible. If $\dim \mathcal{B}(V \oplus W) < \infty$ then for all $s \in O_g$ and $t \in O_h$ we have $(st)^2 = (ts)^2$. 
Since \((st)^2 = (ts)^2\) iff \(t \triangleright (s \triangleright (t \triangleright s)) = s\), this motivated the following definition.

**Definition**

A rack \(X\) is of **Type D** if it contains a decomposable subrack \(Y = R \coprod S\) and elements \(r \in R, s \in S\) such that \(r \triangleright (s \triangleright (r \triangleright s)) \neq s\).

**Theorem [AFGV1]**

A rack \(X\) of type D collapses.

- If \(Z\) is a rack that contains a subrack of type D, respectively projects onto a rack of type D, then \(Z\) is of type D.
- If \(Z\) is indecomposable, then it admits a rack epimorphism \(Z \twoheadrightarrow X\) with \(X\) simple.
**Goal:** Study simple racks. They are classified [AGr], [J]. Among them are the conjugacy classes of finite (non-abelian) simple groups.

- This criterium gives a powerful tool and was succesfully applied for $G = A_n, n \geq 5$ and some sporadic groups.

- Nevertheless, more criteria are needed.

- The study of the structure of $\mathcal{B}(V)$ for $V$ semisimple involved the definition of some reflexions associated to the braided adjoint action, which leaded to the concept of the *Weyl groupoid*.

- After the work of Cuntz & Heckenberger [CH] the following criterium was defined.
On collapsing simple racks

Racks

Criteria: Type F

Definition

A finite rack is of Type F if it has a family of mutually disjoint subracks \((R_a)_{a \in A}\) such that

- \(R_a \triangleright R_b = R_b\) for all \(a, b \in A\);
- for all \(a \neq b \in A\), there are \(r_a \in R_a, r_b \in R_b\) such that \(r_a \triangleright r_b \neq r_b\);
- \(A\) has four elements.

Theorem [ACG1]

A rack \(X\) of type F collapses.

- If \(Z\) is a rack that contains a subrack of type F, respectively projects onto a rack of type F, then \(Z\) is of type F.
On collapsing simple racks

Racks

Criteria: Type C

Using the structure of the Weyl groupoid, recently Heckenberger and Vendramin proved

**Theorem [HV]**

Let $G$ be a non-abelian group, $V, W \in \mathcal{C}_G^G \mathcal{YD}$ simple such that $G$ is generated by $\text{supp}(V \oplus W)$, $\dim V \leq \dim W$ and $(\text{id} - c_W, V c_V, W)(V \otimes W) \neq 0$. Then the following are equivalent:

1. $\dim \mathcal{B}(V \oplus W) < \infty$.
2. $G$, $V$ and $W$ are as in [HV, Theorem 2.1].

In particular, $(\dim V, \dim W)$ belongs to

$$\{(1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}.$$
On collapsing simple racks

Racks

Criteria: Type C

Definition

A rack $X$ is of **Type C** if there are a decomposable subrack $Y = R \sqcup S$ and elements $r \in R$, $s \in S$ such that

- $r \triangleright s \neq s$,
- $R = \mathcal{O}_r^{\text{Inn}} Y$, $S = \mathcal{O}_s^{\text{Inn}} Y$,
- $\min\{|R|, |S|\} > 2$.

If $R$ is indecomposable, then $R = \mathcal{O}_r^{\text{Inn}} R = \mathcal{O}_r^{\text{Inn}} Y$. Moreover, if $R, S$ are indecomposable, then the first condition implies the other two.

Theorem [ACG3]

A rack $X$ of type C collapses.

- If $Z$ is a rack that contains a subrack of type C, respectively projects onto a rack of type C, then $Z$ is of type C.
Example: Consider the conjugacy class $O_x$ in $\text{SL}_3(2)$ with

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then $x$ is an involution, $|O_x| = 21$ and it is not of type D or F. Take

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$ 

We have $y, z \in O_x$ and $x, y, z$ do not commute.

Let $H = \langle x, y, z \rangle \subseteq \text{SL}_3(2)$. It follows that $O^H_x \neq O^H_y$ and a direct computation shows that $O^H_x = \{x, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$ and $O^H_y = O^H_z$.

Thus, $O_x \supseteq O^H_x \sqcup O^H_y$ and $O_x$ is of type C.

Actually, $H \simeq S_4$, $x$ corresponds to $(12)(34)$ and $y, z$ to transpositions.
We say that a rack \( X \) is \( kthulhu \) if it is neither of type D, F or C.

A sober rack is \( kthulhu \).

Table: Some \( kthulhu \) classes in \( \text{PSL}_n(q) \), \( n = 2, 3 \).

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<thead>
<tr>
<th>( n )</th>
<th>( q )</th>
<th>class</th>
<th>type</th>
<th>Remark</th>
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<td>involutions</td>
<td>( kthulhu )</td>
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<td></td>
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<td>sober</td>
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<td></td>
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<td>even or not a square</td>
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<td>unipotent</td>
<td>(2)</td>
<td>( kthulhu )</td>
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<tr>
<td>3</td>
<td>2</td>
<td>unipotent</td>
<td>(3)</td>
<td>( kthulhu )</td>
</tr>
</tbody>
</table>
Ph’nglui mglw’nafh Cthulhu R’lyeh wgah’nagl fhtagn

(In his house at R’lyeh, dead Cthulhu waits dreaming)
H. P. Lovecraft
N. Andruskiewitsch, G. Carnovale, G. A. García.
*Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type I. Non-semisimple classes in* $\text{PSL}_n(q)$, *J. Algebra*, to appear.

N. Andruskiewitsch, G. Carnovale, G. A. García.

N. Andruskiewitsch, G. Carnovale, G. A. García.
*Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type III. Semisimple classes in* $\text{PSL}_n(q)$. *Preprint.*


S. Freyre, M. Graña and L. Vendramin, *On Nichols algebras over* $\text{PGL}(2, q)$ and $\text{PSL}(2, q)$. J. Algebra Appl. 9 (2010), 195–208.


