New Hopf algebras arising from the generalized lifting method

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Rings, modules, and Hopf algebras

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Based on joint work with D. Bagio, J. M. Jury Giraldi and O. Marquez.


Let $\mathbb{k}$ be an algebraically closed field of characteristic zero and let $H$ be a Hopf algebra over $\mathbb{k}$.

As a coalgebra, $H$ has a canonical coalgebra filtration, the *coradical filtration* $\{H_n\}_{n \geq 0}$:

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq \cdots$
- $\bigcup_{n \geq 0} H_n = H$,
- $\Delta(H_n) \subseteq \sum_{i=0}^{n} H_i \otimes H_{n-i}$.

$H_0 = \text{coradical of } H = \sum \text{ of all simple subcoalgebras}.$

$H_n = \bigwedge^{n+1} H_0 = H_{n-1} \wedge H_0.$

$H_n = \{ h \in H : \Delta(h) \in H \otimes H_{n-1} + H_0 \otimes H \}.$

One has that $H_0 = \text{Jac}(H^*)^\perp$ and $H_n = (\text{Jac}(H^*)^{n+1})^\perp$. 
If $H_0$ is a Hopf subalgebra, then the filtration is a Hopf algebra filtration and

$$\text{gr } H = \bigoplus H_n/H_{n-1}, \quad \text{with } H_{-1} = 0$$

is a Hopf algebra.

Take the homogeneous projection

$$\pi : \text{gr } H \rightarrow H_0.$$

It has a Hopf algebra section (the inclusion) and

$$\text{gr } H \simeq R\#H_0 \quad \text{Majid-Radford product or bosonization}$$

here $R = (\text{gr } H)^{\text{co}\pi}$ a braided graded Hopf algebra in $H_0 \mathcal{YD}$.

$H$ is called a lifting of $R$ over $H_0$. 


Let \( V = P(R) = \{ r \in R : \Delta(r) = r \otimes 1 + 1 \otimes r \} \) be the space of primitive elements.

The subalgebra \( \mathcal{B}(V) \) of \( R \) generated by \( V \) is called the **Nichols algebra** of \( V \):

- \( \mathcal{B}(V) \) is graded with \( \mathcal{B}(V)(0) = k \) and \( \mathcal{B}(V)(1) = V \).
- \( \mathcal{B}(V)(1) = P(\mathcal{B}(V)) \).
- \( \mathcal{B}(V) \) is generated by \( V \).

**Rmk:** It is possible to define \( \mathcal{B}(V) \) in terms of the braided vector space \((V, c)\): \( \mathcal{B}(V) = T(V)/J \), with \( J \) the largest two-sided ideal and coideal \( J \subseteq \bigoplus_{n \geq 2} V^n \).
The Lifting Method for fin-dim. Hopf algebras
[Andruskiewitsch-Schneider]

Let $A$ be a finite-dimensional cosemisimple Hopf algebra.

(a) Determine $V \in \mathcal{A} \mathcal{Y} \mathcal{D}$ such that $\mathcal{B}(V)$ is finite-dimensional.

(b) For such $V$, compute all $L$ s.t. $\text{gr } L \simeq \mathcal{B}(V)\#A$.

(c) Prove that for all $H$ such that $H_0 = A$, then $\text{gr } H \simeq \mathcal{B}(V)\#A$.
   (generation in degree one)
Assume $A = k\Gamma$ group algebra over a finite group $\leadsto$ pointed Hopf algebras

- Classification obtained for $\Gamma$ abelian.
- Few examples for $\Gamma$ non-abelian: e.g. $S_3$, $S_4$, $D_{4t}$, $Z_r \rtimes Z_s$.

**Conjecture**

Any finite-dimensional pointed Hopf algebra $H$ s.t. $H_0 \cong k\Gamma$, with $\Gamma$ finite non-abelian simple group is trivial, i.e. $H \cong k\Gamma$.

Verified for $A_n$ with $n \geq 5$, almost all sporadic groups, Suzuki-Ree groups and infinite families of finite simple groups of Lie type.
What if $H_0$ is not a Hopf subalgebra?

[Andruskiewitsch-Cuadra]: replace the coradical filtration by a more general but adequate one $\rightsquigarrow$ the standard filtration $\{H_n\}_{n \geq 0}$

- the subalgebra $H[0]$ of $H$ generated by $H_0$, called the Hopf coradical,

- $H_n = \bigwedge^{n+1} H[0]$.

It holds: If $S$ is bijective then $H[0]$ is a Hopf subalgebra of $H$, $H_n \subseteq H[n]$ and $\{H[n]\}_{n \geq 0}$ is a Hopf algebra filtration of $H$.

In particular,

$$\text{gr } H = \bigoplus_{n \geq 0} H[n]/H[n-1]$$

is a Hopf algebra

If $H_0$ is a Hopf subalgebra, then $H[0] = H_0$ and the coradical filtration coincides with the standard one.
Let $A$ be a finite-dimensional generated by a cosemisimple coalgebra.

(a) Determine $V \in \mathcal{A}YD$ such that $\mathcal{B}(V)$ is finite-dimensional.

(b) For such $V$, compute all $L$ s.t. $\text{gr } L \simeq \mathcal{B}(V)\#A$.

(c) Prove that for all $H$ such that $H_0 = A$, then $\text{gr } H \simeq \mathcal{B}(V)\#A$. (generation in degree one w.r.t. the standard filtration)
**First goal:** Construct new Hopf algebras based on this method.

**First obstruction:** find Hopf algebras generated by their coradicals.

**Source of examples:** quotients of quantum function algebras:

Let $\xi$ be a primitive 4-th root of 1 and let $K$ be generated by $a, b, c, d$ satisfying

\[
ab = \xi ba, \quad ac = \xi ca, \quad 0 = cb = bc, \quad cd = \xi dc, \quad bd = \xi db,
\]
\[
ad = da, \quad ad = 1, \quad 0 = b^2 = c^2, \quad a^2 c = b, \quad a^4 = 1.
\]

The coalgebra structure and its antipode are determined by

\[
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,
\]
\[
\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,
\]
\[
\varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0, \quad \varepsilon(d) = 1
\]
\[
S(a) = d, \quad S(b) = \xi b, \quad S(c) = -\xi c, \quad S(d) = a.
\]
$\mathcal{K}$ is an 8-dimensional Hopf algebra, it is a quotient of $O_q(\text{SL}_2)$, and $\mathcal{K}^*$ is a pointed Hopf algebra $\rightsquigarrow$ basic Hopf algebra.

$\mathcal{K}^* = R_{2,2}$ was first introduced by Radford.

Duals of general Radford algebras $R_{n,m}$ satisfy this property.

**Prop-Def (Andruskiewitsch-Cuadra-Etingof)**

Let $\xi \in \mathbb{G}'_{nm}$. $\mathcal{K}_{n,m} = R^*_{n,m}$ is generated by $U$, $X$ and $A$ satisfying

\[
U^n = 1, \quad X^n = 0, \quad A^m = U,
\]

\[
UX = \omega XU, \quad UA = AU, \quad AX = \xiXA.
\]

As coalgebra $U \in G(\mathcal{K}_{n,m})$, $X \in \mathcal{P}_{1,\mu}(\mathcal{K}_{n,m})$ and

\[
\Delta(A) = A \otimes A + \sum_{k=1}^{n-1} \gamma_{n,k} X^{n-k} U^k A \otimes X^k A
\]

where $\gamma_{n,k} = \frac{1 - \xi^n}{(k)! \omega (n-k)! \omega}$. 


Write $\mathcal{K} = \mathcal{K}_{n,m}$.

**Step (a):**
To describe $V \in \mathcal{K} \mathcal{Y} \mathcal{D}$, we use the equivalence $\mathcal{K} \mathcal{Y} \mathcal{D} \simeq D(\mathcal{K}^{\text{cop}})\mathcal{M}$.

$D(\mathcal{K}^{\text{cop}}) = D$ is a non-semisimple Hopf algebra of tame representation type $\rightsquigarrow$ we describe the simple modules, their projective covers and some indecomposable modules.

For $0 \leq i, j \leq nm - 1$, let $r_{ij} \in \mathbb{N}$ such that $1 \leq r_{ij} \leq n$ and

$$r_{ij} = \begin{cases} i + \frac{j}{m} + 1 \mod n & \text{if } m \mid j, \\ n & \text{if } m \nmid j. \end{cases}$$
New Hopf algebras arising from the generalized lifting method

The Generalized Lifting Method

Step (a) – simple modules

Definition

Let $0 \leq i, j < nm$ and write $r = r_{i,j}$. Let $V_{i,j}$ be the $\mathbb{C}$-vector space with basis $B = \{v_0, \cdots, v_{r-1}\}$ and $D$-action given by

$$A \cdot v_k = \xi^{i-k}v_k \hspace{1cm} g \cdot v_k = \xi^{j-km}v_k \hspace{1cm} \forall \ 0 \leq k \leq r - 1,$$

$$x \cdot v_k = \begin{cases} v_{k+1} & \text{if} \hspace{1cm} 0 \leq k < r - 1, \\ (1 - \xi^{jn})v_0 & \text{if} \hspace{1cm} k = r - 1, \end{cases}$$

$$X \cdot v_k = \begin{cases} 0 & \text{if} \hspace{1cm} k = 0, \\ c_k v_{k-1} & \text{if} \hspace{1cm} 0 < k \leq r - 1, \end{cases}$$

where

$$c_k = (k)\omega \omega^{-k}(\xi^j\omega^{-k+1+i} - 1), \hspace{1cm} \forall \ 1 \leq k \leq r - 1. \hspace{1cm} (1)$$
The Generalized Lifting Method

Step (a) – simple modules / finite-dimensional Nichols algebras

**Theorem (Bagio, G, Jury Giraldi, Marquez)**

\[ \{ V_{i,j} \}_{1 \leq i,j < nm} \text{ is a set of pairwise non-isomorphic simple } \]
\[ D\text{-modules.} \]

**The case } n = 2 = m**

**Theorem (G-Jury Giraldi)**

Let \( M \in \mathcal{K}_Y \mathcal{D} \) be a finite-dimensional non-simple indecomposable module. Then \( \mathcal{B}(M) \) is infinite-dimensional.

**Theorem (G-Jury Giraldi, Xiong, Andruskiewitsch-Angiono)**

Let \( \mathcal{B}(V) \) be a finite-dimensional Nichols algebra over an object \( V \) in \( \mathcal{K}_Y \mathcal{D} \). Then \( V \) is semisimple and isomorphic either to
\[ k_{\chi j} = V_{j,2}, V_{1,j}, V_{2,j}, \bigoplus_{\ell=1}^3 k_{\chi^\ell}, V_{1,j} \oplus k_{\chi}, V_{2,j} \oplus k_{\chi^3}, \]
\[ V_{1,1} \oplus V_{1,3}, V_{2,1} \oplus V_{2,3} \text{ with } j = 1, 3. \]
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Step (a) – Nichols algebras

\[ \mathfrak{B}(\bigoplus_{i=1}^{n} k \chi^i) = \wedge_{i=1}^{n} k \chi^i, \quad \dim \mathfrak{B}(\bigoplus_{i=1}^{n} k \chi^i) = 2^n. \]

\[ \mathfrak{B}(V_{1,j}) = k\langle x, y : x^2 + 2\xi y^2 = 0, xy + yx = 0, x^4 = 0 \rangle, \quad \dim \mathfrak{B}(V_{1,j}) = 8. \text{ The braiding is not diagonal \sim\! new example!} \]

\[ \mathfrak{B}(V_{1,j} \oplus k\chi) = k\langle x, y, z \rangle / J, \text{ with } J \text{ generated by:} \]

\[
\begin{align*}
x^2 + 2\xi y^2 &= 0, \quad xy + yx = 0, \quad x^4 = 0, \quad z^2 = 0, \\
z x^2 + (1 - \xi^j)z x x z - \xi^j x^2 z &= 0, \\
\xi^j x y z - \xi^j x z y + y z x + z x y &= 0, \\
\frac{1}{2}\xi(1 + \xi^{-j})(x z)^2(y z)^2 + (y z)^4 + (z y)^4 &= 0.
\end{align*}
\]

\[ \dim \mathfrak{B}(V_{1,j} \oplus k\chi) = 128. \]
Theorem (G-Jury Giraldi, Xiong, Andruskiewitsch-Angiono)

Let $H$ be a finite-dimensional Hopf algebra over $\mathcal{K}$. Then $H$ is isomorphic either to

(i) $(\bigwedge_{i=1}^{n} \mathbb{k} \chi_{\ell_{i}})^{\#} \mathcal{K}$ with $\ell_{i} = 1, 3$;
(ii) $\mathcal{B}(V_{2,j})^{\#} \mathcal{K}$ for $j = 1, 3$;
(iii) $\mathcal{B}(V_{2,j} \oplus \mathbb{k} \chi_{3})^{\#} \mathcal{K}$;
(iv) $\mathcal{B}(V_{2,1} \oplus V_{2,3})^{\#} \mathcal{K}$
(v) $A_{1,j}(\mu)$ for $j = 1, 3$ and some $\mu \in \mathbb{k}$;
(vi) $A_{1,j,1}(\mu, \nu)$ for $j = 1, 3$ and some $\mu, \nu \in \mathbb{k}$.
(vii) $A_{1,1,1,3}(\mu, \nu)$ for $j = 1, 3$ and some $\mu, \nu \in \mathbb{k}$.
Let \( j \in \{ 1, 3 \} \) and \( \mu \in \mathbb{k} \). The algebra \( A_{1,j} \) is generated by \( a, b, x, y \) satisfying (change \( A = a \) and \( X = b \)):

\[
\begin{align*}
a^4 &= 1, \quad b^2 = 0, \quad ba = \xi ab, \quad ax = \xi xa, \quad bx = \xi xb, \\
ay + ya &= \xi^3 xba^2, \quad by + yb = xa^3, \\
x^4 &= 0, \quad x^2 + 2\xi y^2 = \mu(1 - a^2), \quad xy + yx = \mu\xi^3 ba^3.
\end{align*}
\]

For the coproduct, one has that

\[
\begin{align*}
\Delta(a) &= a \otimes a + \xi^{-1} b \otimes ba^2, \\
\Delta(b) &= b \otimes a^3 + a \otimes b, \\
\Delta(x) &= x \otimes 1 + a^{-j} \otimes x - (1 + \xi^j) ba_1^{-1-j} \otimes y, \\
\Delta(y) &= y \otimes 1 + a^{2-j} \otimes y + \frac{1}{2} \xi(1 - \xi^j) ba_1^{-1-j} \otimes x.
\end{align*}
\]
For Nichols algebras in the general case use techniques of Andruskiewitsch-Angiono to complete Step (a).

Idea:

- \( \mathcal{K}^* = R_{n,m} \) is pointed, i.e. \( \mathcal{K} \) is basic.

- \( R_{n,m} \cong (T_{n,m})_{\sigma} \), the generalized Taft algebra

\[
T_{n,m} = \mathbb{k}\langle g, x : \ x^n = 0, \ g^{nm} = 1, \ gx = \xi^m xg \rangle \\
\cong (\mathbb{k}[x]/(x^n)) \# \mathbb{k}C_{nm} = \mathcal{B}(V) \# \mathbb{k}C_{nm}, \quad V = \mathbb{k}x.
\]

Also, the 2-cocycle \( \sigma \) is known!

- Use the composition of braided monoidal equivalences

\[
F : \quad D \mathcal{M} \xrightarrow{F_1} \mathcal{K} \mathcal{YD} \xrightarrow{F_2} R_{n,m} \mathcal{YD} \xrightarrow{F_3} T_{n,m} \mathcal{YD}
\]
Let $\lambda_{i,j}$ denote a simple object of $\mathcal{C}_{nm} \mathcal{YD}$.

Let $L(\lambda_{i,j})$ be the corresponding simple object in $\mathcal{T}_{nm} \mathcal{YD}$.

**Fact:** It holds that $F(V_{i,j}) = L(\lambda_{-i,-j})$ for all $0 \leq i, j < nm$.

**Theorem (Andruskiewitsch-Angiono)**

Let $V_{i,j}$ be a $D$-simple module. Then $\dim \mathcal{B}(V_{i,j}) < \infty$ if and only if $\dim \mathcal{B}(V \oplus \lambda_{-i,-j}) < \infty$.

**Remark:** $V \oplus \lambda_{-i,-j}$ is a braided vector space of diagonal type $\rightsquigarrow$ we know exactly when $\dim \mathcal{B}(V \oplus \lambda_{-i,-j}) < \infty$.

**Difficult step:** Find the presentation of those $\mathcal{B}(V_{i,j})$ such that $\dim \mathcal{B}(V_{i,j}) < \infty$.

**We have infinite families!!**