GEOMETRY OF EPIMORPHISMS AND FRAMES

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Dedicated to our friend Jorge Solomín

Abstract. Using a bijection between the set $B_H$ of all Bessel sequences in a (separable) Hilbert space $H$ and the space $L(\ell^2, H)$ of all (bounded linear) operators from $\ell^2$ to $H$, we endow the set $F$ of all frames in $H$ with a natural topology for which we determine the connected components of $F$. We show that each component is a homogeneous space of the group $GL(\ell^2)$ of invertible operators of $\ell^2$. This geometrical result shows that every smooth curve in $F$ can be lift to a curve in $GL(\ell^2)$: given a smooth curve $\gamma$ in $F$ such that $\gamma(0) = \Xi$, there exists a smooth curve $\Gamma$ in $GL(\ell^2)$ such that $\gamma = \Gamma \cdot \Xi$, where the dot indicates the action of $GL(\ell^2)$ over $F$.

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1. Introduction

Let $\mathcal{H}$ be a (separable) Hilbert space, $L(\ell^2, \mathcal{H})$ the algebra of all bounded linear operators from $\ell^2$ to $\mathcal{H}$ and $\mathcal{E}$ the subset of $L(\ell^2, \mathcal{H})$ consisting of all epimorphisms from $\ell^2$ onto $\mathcal{H}$. A sequence $\{\xi_n\}$ of elements of $\mathcal{H}$ is called a frame if there exist positive constants $A, B$ such that

$$A\|\xi\|^2 \leq \sum_{n} |\langle \xi, \xi_n \rangle|^2 \leq B\|\xi\|^2 \quad (1.1)$$

for all $\xi \in \mathcal{H}$. Denote by $F$ the set of all frames in $\mathcal{H}$. Frames have been introduced by Duffin and Schaeffer in [10], in connection of nonharmonic Fourier series, but they attract more attention since the beginning of wavelet theory due, in particular, to the fundamental paper [9]. The reader will find many relevant results and facts on frame theory in the book [8] by I. Daubechies, and in several papers, in particular the survey by C. Heil and D. Walnut [12], the monography [11] by D. Han and D. Larson, the exposition [3] of P. Casazza and the survey [4] by O. Christensen. The papers [13] by J. R. Holub and [1] by A. Aldroubi contain some results related to ours. Also, in [2], R. Balan introduces a decomposition in $F$ and defines a metric on each "component" of the partition. In this paper, we proceed in a different way by defining a natural topology in the set $F$ of all frames $\Xi = \{\xi_n\}$ in $\mathcal{H}$. We characterize the connected components of $F$ and show that each component is a homogeneous space of the group $GL(\ell^2)$ of all invertible operators on $\ell^2$. These facts come from the existence of a natural action of $GL(\ell^2)$ over $F$. We get all these results in an indirect way. In fact, we first study the topology of the set $\mathcal{E}$ of all (bounded linear) epimorphisms $\ell^2 \to \mathcal{H}$ and define an action $GL(\ell^2) \times \mathcal{E} \to \mathcal{E}$ to characterize the connected components of $\mathcal{E}$. Then, we define a Banach space $B_{\mathcal{H}}$ of sequences in $\mathcal{H}$ ("Bessel sequences") and a natural isomorphism from $L(\ell^2, \mathcal{H})$

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onto $B_H$ which maps $E$ onto $F$. By means of this isomorphism all facts about $E$ are translated to $F$.

The paper is divided in two parts: in the first part we endow $E$ with the topology induced by the operator norm in $L(\ell^2, H)$. $E$ is an open subset of $L(\ell^2, H)$ and there is a natural action of $GL(\ell^2)$ over $E$, by multiplication on the right.

The orbits of this action are the connected components of $E$. More precisely, for each $n \in \mathbb{N} \cup \{+\infty\}$ the set $\mathcal{E}_n = \{ T \in \mathcal{E} : \dim \ker T = n \}$ is a connected component of $E$ and, as an orbit of the action, it is a homogeneous space of $GL(\ell^2)$. As such, it has several pleasant geometric properties. In particular, continuous (resp. smooth) curves in $\mathcal{E}_n$ lift to continuous (resp. smooth) curves in $GL(\ell^2)$. Thus, any curve $\gamma$ in $\mathcal{E}_n$ has the form $\gamma(t) = T \cdot \Gamma(t)$ for certain curve $\Gamma$ in $GL(\ell^2)$. In the second part of the paper, we observe that the bijection between $F$ and $E$ is the restriction of a natural bijection between $L(\ell^2, H)$ and the space $B_H$ of all Bessel sequences in $H$. It turns out that there is a natural Banach space structure on $B_H$ such that the bijection is an isomorphism of Banach spaces. Thus, the connected components of $F$, which correspond bijectively with the connected components of $E$, are easily determined and the fibration properties of these components allow to characterize their curves by mean of curves in $GL(\ell^2)$.

2. Geometry of epimorphisms

Throughout, $H$ denotes a Hilbert space, $L(H)$ is the algebra of all linear bounded operators on $H$, $L(H)^+$ is the subset of $L(H)$ of all (selfadjoint) positive operators, $GL(H)$ is the group of all invertible operators in $L(H)$ and $GL(H)^+ = GL(H) \cap L(H)^+$ (positive invertible operators). For every $C \in L(H)$ its range is denoted by $R(C)$ and its nullspace by $\ker C$.

Consider two Hilbert spaces $H$ and $K$ and the space $L(H, K)$ of all linear bounded operators from $H$ to $K$. We denote by $E$ the set of all epimorphisms in $L(H, K)$:

$$E = \{ T \in L(H, K) : R(T) = K \}$$

An interesting subset of $E$ is the space $E^o$ of coisometries:

$$E^o = \{ T \in L(H, K) : TT^* = I_K \}.$$

The following result is elementary and will be used frequently:

**Proposition 2.1.** Let $T \in L(H, K)$. Then the following properties are equivalent:

1. $T \in E$
2. $T^*$ is injective and $R(T^*)$ is closed (i.e. $T^*$ is bounded from below).
3. There exists $S \in L(K, H)$ such that $TS = I_K$.
4. $TT^* \in GL(K)$.
5. $TAT^* \in GL(K)^+$ for some (or any) $A \in GL(H)^+$.
6. The transformation $X \mapsto TX$ is an epimorphism from $L(H)$ onto $L(H, K)$.

**Remark 2.2.** Recall the left polar decomposition of $T \in L(H, K)$: there exist $A \in L(H)^+$ and a partial isometry $V \in L(H, K)$ such that $T = VA$. $A$ is uniquely determined by the equation $A = |T| = (T^* T)^{1/2}$ and $V$ is unique provided that $\ker V = \ker T$.

We also have the right polar decomposition of $T \in L(H, K)$: there exist $B \in L(K)^+$ and a partial isometry $W \in L(H, K)$ such that $T = BW$. $B$ is uniquely determined by the equation $B = |T^*| = (TT^*)^{1/2}$ and $W$ is unique provided that $\ker W = \ker T$.

With these facts and notations, we can add two different equivalent conditions to the list of Proposition 2.1:

1. $B \in GL(K)$.
2. $W \in E^o$, i.e., $W$ is a coisometry.

**Corollary 2.3.** $E$ is open in $L(H, K)$ and $E^o$ is closed in $E$. 

Proof. Consider the continuous map \( \alpha : L(\mathcal{H}, \mathcal{K}) \to L(\mathcal{K}) \) given by \( \alpha(T) = TT^* \), \( T \in L(\mathcal{H}, \mathcal{K}) \). Then \( \mathcal{E} = \alpha^{-1}(\text{GL}(\mathcal{H})) \) and \( \mathcal{E}^o = \alpha^{-1}(\{I\}) \). \( \square \)

The following observation will be used later.

**Proposition 2.4.** \( \mathcal{E}^o \) is strong deformation retract of \( \mathcal{E} \).

Proof. Consider the map \( \rho : \mathcal{E} \to \mathcal{E}^o \) given by \( \rho(T) = (TT^*)^{-1/2}T \). Clearly \( \rho \) is a continuous retraction. For \( t \in [0, 1] \), the maps

\[
\rho_t(T) = (TT^*)^{-t/2}T, \quad T \in \mathcal{E}
\]

define a deformation between \( 1_\mathcal{E} \) and \( \rho \).

If \( T \in \mathcal{E}^o \) then it is easy to see that \( TT^* = I_K \) and \( T^*T = P_{(\ker T)^\perp} = P_{(T^*)^\perp} \). As a consequence, we get that the map \( \theta : \mathcal{E} \to \mathcal{P} \) given by \( \theta(T) = P_{(\ker T)^\perp} \) is continuous. Moreover:

**Proposition 2.5.** If \( T \in \mathcal{E} \), the Moore-Penrose pseudoinverse of \( T \) is \( T^\dagger = T^*(TT^*)^{-1}T \) and \( P_{(\ker T)^\perp} = T^*(TT^*)^{-1}T \).

Proof. Let \( X = T^*(TT^*)^{-1} \). Then \( TX = I, XTX = X \) and \( TXT = T \), so that \( X \) is a pseudoinverse of \( T \) and \( XT, XT^* \) are (not necessarily orthogonal) projections. On the other hand, \( TX = I \) and \( XT^* = T^*(TT^*)^{-1}T \) which are clearly selfadjoint. Therefore \( X = T^\dagger \) and \( P_{(\ker T)^\perp} = T^1\dagger T = T^*(TT^*)^{-1}T \). \( \square \)

**Corollary 2.6.** The maps \( \mu : \mathcal{E} \to L(\mathcal{K}, \mathcal{H}) \) and \( \nu : \mathcal{E} \to L(\mathcal{H}) \) defined by \( \mu(T) = T^\dagger \) and \( \nu(T) = P_{(\ker T)^\perp} \), respectively, are real analytic.

2.1. **The action of GL(\mathcal{H}) on \mathcal{E}**. Consider the following left action of \( \text{GL}(\mathcal{H}) \) on \( \mathcal{E} \):

\[
\text{GL}(\mathcal{H}) \times \mathcal{E} \to \mathcal{E}, \quad (V, T) \mapsto TV^{-1}.
\]

The orbit of \( T \in \mathcal{E} \) by this action is the set \( T \cdot \text{GL}(\mathcal{H}) \).

**Theorem 2.7.** Let \( T \in \mathcal{E} \). Then the orbit \( T \cdot \text{GL}(\mathcal{H}) \) is open and it is the connected component of \( T \) in \( \mathcal{E} \).

Proof. Since \( \text{GL}(\mathcal{H}) \) is open and connected in \( L(\mathcal{H}) \) and \( X \mapsto TX \) is continuous and linear from \( L(\mathcal{H}) \) onto \( L(\mathcal{H}, K) \), it holds that \( T \cdot \text{GL}(\mathcal{H}) \) is open (by the open mapping theorem) and connected in \( L(\mathcal{H}, K) \). \( \square \)

For each \( n \in \mathbb{N} \cup \{\infty\} \), define the sets

\[
\mathcal{E}_n = \{T \in \mathcal{E} : \dim \ker T = n\} \quad \text{and} \quad \mathcal{E}^o_n := \{S \in \mathcal{E}^o : \dim \ker S = n\}.
\]

So that

\[
\mathcal{E} = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{E}_n \quad \text{and} \quad \mathcal{E}^o = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{E}^o_n.
\]

We prove that the connected components of \( \mathcal{E} \) (resp. \( \mathcal{E}^o \)) are, precisely, \( \mathcal{E}_n \) (resp. \( \mathcal{E}^o_n \)).

**Proposition 2.8.** Let \( n \in \mathbb{N} \cup \{\infty\} \), then

1. Given \( T_1, T_2 \in \mathcal{E}_n \), there exists \( V \in \text{GL}(\mathcal{H}) \) such that \( T_2 = T_1V \). In other words, if \( T \in \mathcal{E}_n \), then \( \mathcal{E}_n = T \cdot \text{GL}(\mathcal{H}) \).
2. Given \( T_1, T_2 \in \mathcal{E}^o_n \), there exists \( U \in \mathcal{U}(\mathcal{H}) \) such that \( T_2 = T_1U \). In other words, if \( T \in \mathcal{E}^o_n \), then \( \mathcal{E}^o_n = T \cdot \mathcal{U}(\mathcal{H}) = \{TU^o : U \in \mathcal{U}(\mathcal{H})\} \).

Proof. (1) The operator \( V_1 = T_1^JT_2 : \ker T_2 \to \ker T_1^\perp \) is invertible. It can be "completed" to an invertible operator \( V = V_1 + V_2 \in \text{GL}(\mathcal{H}) \), choosing any isomorphism \( V_2 : \ker T_2 \to \ker T_1 \). It is clear that \( T_1V = T_1V_1 = T_2 \).
(2) It follows the same lines, but $V_1$ and $V_2$ are unitaries.

Proposition 2.9. Let $T \in \mathcal{E}$. Then the mapping
\[
\tau_T : GL(\mathcal{H}) \to \mathcal{E}, \quad \tau_T(V) = TV^{-1}
\]
admits analytic local cross sections.

Proof. We must prove that for every $T \in \mathcal{E}$ there exists a neighborhood $B$ of $T$ in $\mathcal{E}$ and an analytic map $\sigma : B \to GL(\mathcal{H})$ such that $\tau_T(\sigma(T')) = T'$ for all $T' \in B$. Choose $S \in L(\mathcal{K}, \mathcal{H})$ such that $TS = I_\mathcal{K}$. Taking $\varepsilon = \|S\|^{-1}$, if $W \in L(\mathcal{H}, \mathcal{K})$ and $\|T - W\| < \varepsilon$, then $\|I_\mathcal{K} - WS\| \leq \|T - W\| \|S\| < 1$. Hence $WS \in GL(\mathcal{K})$ and $W \in \mathcal{E}$. Also $SW + (I_\mathcal{H} - ST) \in GL(\mathcal{H})$, because $\|SW + (I_\mathcal{H} - ST) - I_\mathcal{H}\| \leq \|S\| \|W - T\| < 1$. For $W \in L(\mathcal{H}, \mathcal{K})$ such that $\|T - W\| < \varepsilon$, define
\[
\sigma(W) = (SW + (I_\mathcal{H} - ST))^{-1}.
\]
Then the map $\sigma$ is analytic and it is a local cross section of $\tau_T$, because $\tau_T(\sigma(W)) = T\sigma(W)^{-1} = T(SW + (I_\mathcal{H} - ST)) = W$. □

Corollary 2.10. Let $T \in \mathcal{E}$ and $n = \dim \ker T$. Then $\mathcal{E}_n = T \cdot GL(\mathcal{H})$ is homeomorphic to the homogeneous space $GL(\mathcal{H})/\mathcal{I}_T$, where $\mathcal{I}_T$ is the isotropy group at $T$ of the action of $GL(\mathcal{H})$ on $\mathcal{E}$, i.e.
\[
(2.1) \quad \mathcal{I}_T = \{V \in GL(\mathcal{H}) : TV = T\}.
\]

Proposition 2.11. Let $T \in \mathcal{E}$ and denote by $P = T^\dagger T = P_{\ker T^\perp}$. Then the isotropy group $\mathcal{I}_T$ of $T$, defined in (2.1), can be characterized in the matrix representation of $L(\mathcal{H})$ given by $P$, by
\[
(2.2) \quad \mathcal{I}_T = \{ \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} : y \in GL(\ker T) \}.
\]

Proof. The matrix form of $T$ is $T = \begin{pmatrix} T_1 & 0 \\ T_2 & 0 \end{pmatrix}$ with $T_1^\dagger T_1 + T_2^\dagger T_2 \in GL(\ker T^\perp)^\dagger$.

If $V = \begin{pmatrix} a & b \\ x & y \end{pmatrix} \in GL(\mathcal{H})$, then
\[
(2.3) \quad TV = T \iff \begin{pmatrix} T_1 a & T_1 b \\ T_2 a & T_2 b \end{pmatrix} = T.
\]
This shows that if $a = 1$ and $b = 0$, then $V \in \mathcal{I}_T$. On the other hand, if $TV = T$, then by equation (2.3),
\[
T^* T = \begin{pmatrix} T_1^* T_1 + T_2^* T_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^*)^*(T_1^* T_1 + T_2^* T_2) & * \\ * & b^*(T_1^* T_1 + T_2^* T_2) \end{pmatrix},
\]
showing that $a = 1$ and $b = 0$. Finally, the fact that $y \in GL(\ker T)$ is equivalent to the fact that $V \in GL(\mathcal{H})$. □

Remark 2.12. Fix $n \in \mathbb{N} \cup \{\infty\}$. $\mathcal{E}_n$ has a natural structure of analytic submanifold of $L(\mathcal{H}, \mathcal{K})$ as an open subset. By equation (2.2) $\mathcal{I}_T$ is a regular Lie-Banach subgroup of $GL(\mathcal{H})$ and by Proposition 2.9 the map $\tau_T : GL(\mathcal{H}) \to \mathcal{E}_n$ is open. Therefore, the above mentioned is the unique structure of differentiable manifold of $\mathcal{E}_n$ which makes the map $\tau_T$ a submersion. This means that the homeomorphism of Corollary 2.10 becomes a diffeomorphism.
Remark 2.13. A particularly interesting local cross section for \( \tau T \) is
\[
(2.4) \quad \sigma^1(W) = \left( T^\dagger W + (1 - T^\dagger)T \right)^{-1} = \left( T^\dagger W + P_{\ker T} \right)^{-1},
\]
defined for \( W \in L(H,K) \) such that \( \|T - W\| < \|T^\dagger\|^{-1} \). The advantage of this section over the one defined in the proof of Proposition 2.9 is that the map
\[(T, W) \mapsto \sigma^1(W)\]
is real analytic in both variables.

**Theorem 2.14.** Consider the map \( \alpha : E \to GL(K)^+ \) given by \( \alpha(T) = TT^* \) (\( T \in E \)). Then for every \( T \in E \) it holds
\[
\alpha(T \cdot GL(H)) = GL(K)^+.
\]

**Proof.** First we prove that \( \{TAT^* : A \in GL(H)^+ \} = GL(K)^+ \): if \( B \in GL(K)^+ \), then \( A = T^\dagger B(T^\dagger)^* + P_{\ker T} \in GL(H)^+ \) (because \( R(T^\dagger) = (\ker T^\dagger)^\perp \)) and it satisfies \( TAT^* = B \). The reverse inclusion follows from Proposition 2.1.

Now, \( \alpha(T \cdot GL(H)) = \{TVV^*T^* : V \in GL(H) \} = \{TAT^* : A \in GL(H)^+ \} \). \( \square \)

**Corollary 2.15.** The mapping \( \alpha : E \to GL(K)^+ \) is a splitting bundle with fibres \( \alpha^{-1}(A) = A^{1/2}E^o \). Moreover, for every \( n \in \mathbb{N} \cup \{0, \infty\} \), \( \alpha|_{\mathcal{E}_n} : \mathcal{E}_n \to GL(K) \) is a splitting bundle with global cross section \( \sigma(A) = A^{1/2}T_n \), for a fixed \( T_n \in \mathcal{E}_n \).

We are interested in the fibres of the bundle \( \alpha_n : \mathcal{E}_n \to GL(K)^+ \), given by \( \alpha_n = \alpha|_{\mathcal{E}_n} \), i.e. \( \alpha_n(T) = TT^*, T \in \mathcal{E}_n \). Fix \( S \in GL(K)^+ \) and \( T \in \mathcal{E}_n \) such that \( TT^* = S \), i.e. \( T \in \alpha_n^{-1}(S) \). Clearly, for every \( U \in U(H) \), also \( TU^* \in \alpha_n^{-1}(S) \). Moreover:

**Proposition 2.16.** Let \( S \in GL(K)^+ \) and \( T \in \mathcal{E}_n \) such that \( TT^* = S \). Then
\[
(2.5) \quad \alpha_n^{-1}(S) = T \cdot U(H) = \{TU^* : U \in U(H) \}.
\]

**Proof.** By Corollary 2.15, \( \alpha_n^{-1}(S) = S^{1/2}E^o_n \). By Proposition 2.8, given \( V_1, V_2 \in E_n^o \), there exists \( U \in U(H) \) such that \( V_1U^* = V_2 \), showing formula (2.5). \( \square \)

**Remark 2.17.** Since the fibration \( \alpha_n : \mathcal{E}_n \to GL(K)^+ \) splits by means of the global cross section defined in Corollary 2.15, it follows that, for every fixed \( T \in \mathcal{E}_n \), then \( \mathcal{E}_n \) is diffeomorphic to \( GL(K)^+ \times T \cdot U(H) \). The geometry of the space \( GL(K)^+ \) is very well studied, see [7], [5]. The study of the geometry of the fibre \( T \cdot U(H) \) (which is also an orbit) will be done elsewhere.

**Splitting curves.** Fix \( T \in \mathcal{E}_n \). Recall that the space \( \mathcal{E}_n \) is open in \( L(H,K) \) and \( \mathcal{E}_n = T \cdot GL(H) \) is the orbit of \( T \) by the action of \( GL(H) \). We shall describe now the geometry of \( \mathcal{E}_n \). The proofs of the statements of this section appear in [7]. Denote by
\[
\mathcal{S}_n = \{(T, S) \in L(H,K) \times L(K,H) : T \in \mathcal{E}_n, TS = I\}.
\]
This space has a rich geometrical structure by the action of \( GL(H) \) given by
\[
W \cdot (T, S) = (TW^{-1}, WS), \quad W \in GL(H).
\]
In fact, for any fixed pair \( (T, S) \in \mathcal{S}_n \), the map \( \tau : GL(H) \to \mathcal{S}_n, \tau(W) = W \cdot (T, S) \) defines an homogeneous reductive space with a connection given by the distribution of horizontal spaces. Note that the map \( \tau : \mathcal{S}_n \to \mathcal{E}_n, \tau(T, S) = T \) defines a fibre bundle with affine fibres.

Given a smooth curve \( \gamma : [0, 1] \to \mathcal{S}_n \) such that \( \gamma(0) = (T, S), \gamma(t) = (a(t), b(t)) \), the unique solution of the differential equation
\[
\begin{align*}
\dot{\Gamma} & = \dot{a}b - ab(1 - ab)
\Gamma \\
\Gamma(0) & = I
\end{align*}
\]
satisfies that \( \Gamma(t) \in GL(\mathcal{H}) \), \( \Gamma(t) \cdot (T,S) = \gamma(t) \). Consider now a smooth curve \( \delta : [0,1] \to \mathcal{E}_n \) and define
\[
\gamma(t) = (\delta(t), \delta(t)^\dagger) = (\delta(t), \delta(t)^* \delta(t)^* \delta(t)^\dagger)^{-1}.
\]
Observe that \( \gamma(t) \in \mathcal{S}_n \) \( \forall t \) and \( \gamma(0) = (\delta(0), \delta(0)^\dagger) \). Then \( \delta(t) = \delta(0) \Gamma(t)^{-1} \).

**Remark 2.18.** In [2], R. Balan implicitly studies the following action of \( GL(\mathcal{H}) \) over \( \mathcal{E} \):
\[
GL(\mathcal{H}) \times \mathcal{E} \to \mathcal{E} \quad (V,T) \mapsto VT.
\]
This action is free: \( V_1 T = V_2 T \) only if \( V_1 = V_2 \). The orbit of \( T \in \mathcal{E} \) under this action is much smaller that the orbit under the action we considered. In fact, Balan proves that \( T_1 \in \mathcal{E} \) belongs to the orbit of \( T \) if and only if \( R(T_1 T_1^*) = R(T^* T) \). However, under the action \( GL(\ell^2) \times \mathcal{E} \to \mathcal{E} \), \( (W,T) \mapsto TW^{-1} \), \( T_1 \in \mathcal{E} \) belongs to the orbit of \( T \) if and only if \( T_1 T_1^* \) is congruent to \( T^* T \), in the sense that there exists \( W \in GL(\ell^2) \) such that \( W^* (T_1 T_1^*) W = T^* T \). Of course, this condition does not imply that \( R(T_1 T_1^*) = R(T^* T) \). The converse, however, is true. The relevant fact about Balan’s orbits is that there is a natural metric defined on each orbit and he uses this metric to find, given a frame \( \Xi \), its closest tight frame. We shall study metrics in our orbits elsewhere.

**Remark 2.19.** Let \( CR_n^+(\mathcal{H}) \) denote the set of all positive (semidefinite) closed range operators \( A \) on \( \mathcal{H} \) such that \( \dim \ker A = n \).

From some results obtained in [6], where the congruence orbits of any positive operator is studied, it follows that the map
\[
\beta_n : \mathcal{E}_n \to CR_n^+(\mathcal{H}), \quad \beta_n(T) = T^* T
\]
has continuous local cross sections for \( n \in \mathbb{N} \cup \{0\} \). The result fails if \( n = \infty \). Analogous results hold for the maps \( \mu_n : CR_n^+(\mathcal{H}) \to CR_n^+(\mathcal{H}), \mu_n(A) = A^\dagger \) and \( \nu_n : CR_n^+(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \nu_n(A) = P_{\ker A}^\perp \). The fact about \( \beta_n \) allows the study of \( \mathcal{E}_n \) as the total space of a fibre bundle over \( CR_n^+(\mathcal{H}) \). It should be mentioned that the geometry of \( CR_n^+(\mathcal{H}) \) is well known [6], so that the fibration properties of \( \beta_n \) may be of great help in order to completely understand the geometry of \( \mathcal{E}_n \).

3. Frames

Consider a sequence \( \Xi = (\xi_n)_{n \in \mathbb{N}} \) in a Hilbert space \( \mathcal{H} \); \( \Xi \) is called a **Bessel sequence** if there exists a positive number \( B \) such that
\[
\sum_{n=1}^{\infty} |\langle \xi, \xi_n \rangle|^2 \leq B||\xi||^2, \quad \xi \in \mathcal{H}.
\]

**Proposition 3.1.** For a sequence \( \Xi = (\xi_n) \) in a Hilbert space \( \mathcal{H} \), the following are equivalent:

1. \( \Xi \) is a Bessel sequence
2. there is a bounded linear operator \( W : \mathcal{H} \to \ell^2 \) such that \( \quad W\xi = (\langle \xi, \xi_n \rangle)_{n \in \mathbb{N}}, \quad \xi \in \mathcal{H} \)
3. there is a linear operator \( T : \ell^2 \to \mathcal{K} \) such that \( T \epsilon_n = \xi_n \), where \( \epsilon_n \) denotes the \( n \)-th vector of the canonical orthonormal basis of \( \ell^2 \).

The proof is straightforward. Observe that, if \( \Xi \) is a Bessel sequence, then \( ||W\xi||_2 \leq B^{1/2}||\xi||_\mathcal{H} \) for all \( \xi \in \mathcal{H} \). In this case, \( T = W^* \). As a corollary, the set \( \mathcal{B}_\mathcal{H} \) of Bessel
sequences in $\mathcal{H}$ is a $\mathbb{C}$-vector space. Moreover, if we define
\[
\|\Xi\|_B = \inf \{B^{1/2} : \sum |\langle \xi, \xi_n \rangle|^2 \leq B\|\xi\|^2, \xi \in \mathcal{H} \}
\]
\[
= \sup \{(\sum |\langle \xi, \xi_n \rangle|^2)^{1/2} : \xi \in \mathcal{H}, \|\xi\| \leq 1 \}
\]
\[
= \sup \{\|\sum_{n=1}^{\infty} a_n \xi_n\| : (a_n) \in \ell^2, \|(a_n)\|_2 = 1 \},
\]
then $(\mathcal{B}_\mathcal{H}, \|\cdot\|_B)$ is a Banach space isometrically isomorphic to $L(\ell^2, \mathcal{H})$: the isomorphism maps a Bessel sequence $\Xi = (\xi_n)$ into the operator $T \in L(\ell^2, \mathcal{H})$ defined by
\[
T_\Xi((a_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} a_n \xi_n, \ (a_n)_{n\in\mathbb{N}} \in \ell^2;
\]
the inverse isomorphism maps $T \in L(\ell^2, \mathcal{H})$ into $\Xi_T = (Te_n)$.

A Bessel sequence $\Xi = (\xi_n)$ is called a frame in $\mathcal{H}$ if there exist positive constants $A, B$ such that
\[
A\|\xi\|^2 \leq \sum |\langle \xi, \xi_n \rangle|^2 \leq B\|\xi\|^2, \ \xi \in \mathcal{H}.
\]
Observe that this condition together with equation (3.1) is equivalent to
\[
A\|\xi\|^2 \leq \langle T_\Xi T_\Xi^* \xi, \xi \rangle \leq B\|\xi\|^2, \ \xi \in \mathcal{H},
\]
or, what is the same,
\[
AI_{\mathcal{H}} \leq T_\Xi T_\Xi^* \leq BI_{\mathcal{H}}.
\]
Of course, this means that $T_\Xi$ is a epimorphism from $\ell^2$ onto $\mathcal{H}$ or, equivalently, that $T_\Xi^* \in L(\mathcal{H}, \ell^2)$ is bounded from below. Thus the isomorphism $\Theta : L(\ell^2, \mathcal{H}) \to \mathcal{B}_\mathcal{H}$, $\Theta(T) = (Te_n)$ maps the set of epimorphisms in $L(\ell^2, \mathcal{H})$ onto the set $\mathcal{F}_\mathcal{H}$ of all frames in $\mathcal{H}$. Observe that the positive invertible $T_\Xi T_\Xi^* \in L(\mathcal{H})$ is usually called the frame operator of $\Xi$. $T_\Xi$ is called the synthesis operator of $\Xi$ and $T_\Xi^*$ is called the analysis operator of $\Xi$ ([14], [3]).

A frame $\Xi = (\xi_n)$ is called tight if there exists $A > 0$ such that
\[
\sum |\langle \xi, \xi_n \rangle|^2 = A\|\xi\|^2, \ \xi \in \mathcal{H}.
\]
This means that $T_\Xi T_\Xi^* = AI_{\mathcal{H}}$, so that the set $\mathcal{F}^t_\mathcal{H}$ of tight frames in $\mathcal{H}$ corresponds (under the isomorphism $\Theta$) with the set $\mathbb{R}^+ \mathcal{E}_0$ of positive scalar multiples of cosoiometries from $\mathcal{H}$ into $\ell^2$.

A frame $\Xi = (\xi_n)$ is called exact if no proper subsequence of $\Xi$ is a frame. It is known ([15], [10]) that this is equivalent to $(\xi_n)$ being a Riesz basis or, what is the same, to $T_\Xi$ being invertible.

There is a natural action of $GL(\ell^2)$ over $\mathcal{F}_\mathcal{H}$. In fact, given $\Xi \in \mathcal{F}_\mathcal{H}$ and $V \in GL(\ell^2)$, define $V \cdot \Xi = ((T_\Xi \circ V^{-1})e_n)$. In terms of the matrix $A = (a_{nm})$, where $a_{nm} = \langle V^{-1}e_n, e_m \rangle$, $V \cdot \Xi$ is defined as the (formal) product $A\Xi$, i.e., $V \cdot \Xi = (\eta_n)$ where
\[
\eta_n = \sum_{m=1}^{\infty} a_{nm}\xi_m.
\]
This action corresponds bijectively with that of $GL(\ell^2)$ over $\mathcal{E}$, so that the orbits are the connected components of $\mathcal{F}_\mathcal{H}$. The next result collect similar facts on $\mathcal{F} = \mathcal{F}_\mathcal{H}$ to those proved for $\mathcal{E}$. The proof follows from the fact that $\Theta$ is an isomorphism.

**Theorem 3.2.** Let $\mathcal{H}$ be a (separable) Hilbert space.

(1) $\mathcal{F}$ is an open subset of $\mathcal{B}_\mathcal{H}$, so that the connected components are arcwise connected.
Any connected component of \( F_\mathcal{H} \) has the form
\[
F_n = \{ \Xi \in F : \dim \ker T\Xi = n \}.
\]
for \( n \in \mathbb{N} \cup \{\infty\} \). In particular, the set of Riesz basis of \( \mathcal{H} \) is arcwise connected.

If \( \Xi = (\xi_n) \in F_n \), then any other \( E = (\eta_n) \in F_n \) has the form \( E = V \cdot \Xi \) for some \( V \in \text{GL}(\ell^2) \).

For any \( \Xi \in F_n \) the map \( \text{GL}(\ell^2) \to F_n \), defined by \( V \mapsto V \cdot \Xi \), is a homogeneous space with isotropy group \( G_\Xi = \{ V \in \text{GL}(\ell^2) : V \cdot \Xi = \Xi \} \); analogously, \( \mathcal{U}(\ell^2) \to F_\mathcal{H}^\circ \), defined by \( U \mapsto U \cdot \Xi \), is a homogeneous space with isotropy group \( U_\Xi = \{ U \in \mathcal{U}(\ell^2) : U \cdot \Xi = \Xi \} \).

Any continuous (resp. differentiable) curve \( \gamma \) in \( F_n \) such that \( \gamma(0) = \Xi \) has the form \( t \mapsto \Gamma(t) \cdot \Xi \) for some continuous (resp. differentiable) curve \( \Gamma \) in \( \text{GL}(\ell^2) \). Analogously, any curve \( \gamma \) in \( F_n^\circ \) with \( \gamma(0) = \Xi \) has the form \( \gamma(t) = \Gamma(t) \cdot \Xi \) where \( \Gamma \) is a curve in \( \mathcal{U}(\mathcal{H}) \).

\( F_n^\circ \) is a deformation retract of \( F_n \) for all \( n \).

Remark 3.3. The separability hypothesis is not an essential one. In fact, all results can be proven in the general sense, using minor changes. On the other hand, the results of the paper can easily be generalized to the setting of frames in Hilbert C*-modules.

References


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