Generalized Schur Complements and P-complementable Operators

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Abstract

Let $A$ be a selfadjoint operator and $P$ be an orthogonal projection both operating on a Hilbert space $\mathcal{H}$. We say that $A$ is $P$-complementable if $A - \mu P \geq 0$ holds for some $\mu \in \mathbb{R}$. In this case we define $I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}$. As a tool for computing $I_P(A)$ we introduce a natural generalization of the Schur complement or shorted operator of $A$ to $\mathcal{S} = R(P)$, denoted by $\Sigma(A, P)$. We give expressions and a characterization for $I_P(A)$ that generalize some known results for particular choices of $P$. We also study some aspects of the shorted operator $\Sigma(A, P)$ for $P$-complementable $A$, under the hypothesis of compatibility of the pair $(A, \mathcal{S})$. We give some applications in the finite dimensional context.

Key words: Positive semidefinite operators, shorted operator, Hadamard product, completely positive maps.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Given a closed subspace $S \subseteq \mathcal{H}$ and $P = P_S \in L(\mathcal{H})$ the orthogonal projection onto $S$, we study the following two problems: for any selfadjoint operator $A \in L(\mathcal{H})$,

1. determine whether there exists some $\mu \in \mathbb{R}$ such that
   \[ A - \mu P \geq 0; \quad (1) \]

2. in case equation (1) holds for some $\mu$, compute the optimum number
   \[ I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}. \quad (2) \]

The general solution of problem (1) is well known, see for example Pekarev [14] or Proposition 3.3 below. Also, if $A \geq 0$, problem (2) has a known answer (see, for example, [7]). We state this case in the preliminary section 2 (Corollary 2.2). Therefore our main interest is to study problem (2) in the non-positive case. It should be mentioned that the general case seems not to be easily reduced to the positive case (see Remark 5.1).

If condition (1) is satisfied by $A$, we shall say that $A$ is $P$-complementable, because in this case there exists the shorted operator (or Schur complement, see Anderson-Trapp [1]) defined as follows:

\[ \Sigma(A,P) = \max\{D \in L(\mathcal{H}) : D = D^*, \; D \leq A, \; D(\mathcal{H}) \subseteq S\}. \]

Using the identity $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$, in Section 3 we extend several known properties of shorted operators of positive operators to our case. On the other hand, in section 5 we show that, if $A \not\geq 0$ but it is $P$-complementable, then

\[ I_P(A) = \lambda_{\min}(\Sigma(A, P)), \]

where $\lambda_{\min}(C)$ denotes the minimum of the spectrum $\sigma(C)$ of $C \in L(\mathcal{H})$.

Although most applications of the problems mentioned above appear in matrix theory, i.e. when $\dim \mathcal{H} < \infty$, an additional hypothesis of the operator $A$ allows to extend all finite dimensional results to our setting. This hypothesis is the so called compatibility of the pair $(A, S)$. This notion, defined by G. Corach, A. Maestripieri and the second author in [4], [5], [6], is the following: The pair $(A, S)$ is compatible if there exists a $A$-selfadjoint projection onto $S^\perp$, i.e., $Q \in L(\mathcal{H})$ such that $Q^2 = Q$, $AQ = Q^*A$ and $R(Q) = S^\perp$.

There are several characterization of the compatibility of $(A, S)$ and general properties of such pairs in the case $A \geq 0$ (see, for example, [4]); some of them are stated in Section 4 of this paper, where we also extend these properties to the non-positive case.
We say that compatibility is an additional condition because, if \((1 - P)A(1 - P) \geq 0\) and \((A, S)\) is compatible, then \(A\) is \(P\)-complementable. The reverse implication is false in general, but it is true if \(\dim S^\perp < \infty\), in particular in the finite dimensional case.

Section 5 is devoted to the computation of the number \(I_P(A)\) for \(A\) selfadjoint, not necessarily positive. We first obtain the formula

\[
I_P(A) = \inf\{\langle A\xi, \xi \rangle : \xi \in \mathcal{H}, \|P\xi\| = 1\}
\]

so that, if \(S\) is the subspace generated by the unit vector \(\xi \in \mathcal{H}\), then

\[
I_P(A) = \inf\{\langle A\eta, \eta \rangle : \eta \in \mathcal{H}, \langle \eta, \xi \rangle = 1\}.
\]

If \((A, S)\) is compatible, we show that the computation of \(I_P(A)\) can be reduced to the case in which \(S \subseteq \overline{R(A)}\), by replacing \(S\) by \(\overline{S \cap R(A)}\). We state the results of the rest of this section in the following theorem:

**Theorem.** Let \(A = A^* \in L(\mathcal{H})\), \(A \not\geq 0\), and \(P = P^* = P^2 \in L(\mathcal{H})\) with \(R(P) = S\). Suppose that \((1 - P)A(1 - P) \geq 0\) and \((A, S)\) is compatible. Then

1. \(R(\Sigma(A, P)) = S \cap R(A) \neq \{0\}\).
2. If \(T = \overline{S \cap R(A)}\) and \(Q = P_T\), then the pair \((A, T)\) is compatible, \(\Sigma(A, P) = \Sigma(A, Q)\) and \(I_P(A) = I_Q(A)\).
3. If \(R(A)\) is closed, then \(T = \overline{S \cap R(A)}\) and

\[
I_P(A) = I_Q(A) = \lambda_{\text{min}}((QA^\dagger Q)^\dagger), \tag{3}
\]

where \(C^\dagger\) denotes the Moore-Penrose pseudoinverse of a closed range operator \(C\).

Formula (3) is the natural generalization of \(I_P(A) = \|PA^\dagger P\|^{-1}\), which holds if \(A\) is positive (semidefinite) with closed range, see Corollary 2.2. In section 6 we study some applications of the mentioned results, particularly to problems posed by M. Fiedler-T.L. Markham [9] and R. Reams [15]. Given a completely positive map \(\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})\), we also compute the number

\[
I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu \cdot \text{Id} \text{ is completely positive}\},
\]

which can be considered as a notion of index for such maps.
2 Preliminary results

In this paper $\mathcal{H}$ denotes a Hilbert space, $L(\mathcal{H})$ is the algebra of all linear bounded operators on $\mathcal{H}$, $Gl(\mathcal{H})$ is the group of invertible operators in $L(\mathcal{H})$ and $L(\mathcal{H})^+$ is the subset of $L(\mathcal{H})$ of all positive (semidefinite) operators. If dim $\mathcal{H} = n < \infty$ we shall identify $\mathcal{H}$ with $\mathbb{C}^n$ and $L(\mathcal{H})$ with the space of $n \times n$ complex matrices $M_n(\mathbb{C})$. The elements of $\mathbb{C}^n$ are considered as column vectors. For simplicity we sometimes describe a column vector $\xi \in \mathbb{C}^n$ as $\xi = (\xi_1, \ldots, \xi_n)$.

For every $C \in L(\mathcal{H})$ its range is denoted by $R(C)$, $\sigma(C)$ denotes the spectrum of $C$ and $\rho(C)$ the spectral radius of $C$. If $R(C)$ is closed, then $C^\dagger$ denotes the Moore-Penrose pseudoinverse of $C$. The orthogonal projection onto a closed subspace $S$ is denoted by $P_S$. We use the notations $Q = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$ for the set of idempotents and $P = \{P \in Q : P = P^*\}$ for the set of orthogonal projections. For every $P \in P$, the decomposition $\mathcal{H} = R(1 - P) \oplus R(P)$ induces a $2 \times 2$ representation of $A \in L(\mathcal{H})$:

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

which we call the matrix representation induced by $P$.

Now we state the well known criterion due to Douglas [8] (see also Fillmore-Williams [10]) about ranges and factorization of operators:

**Theorem 2.1.** Let $A, B \in L(\mathcal{H})$. Then the following conditions are equivalent:

1. $R(B) \subseteq R(A)$.
2. There exists a positive number $\mu$ such that $BB^* \leq \mu AA^*$.
3. There exists $D \in L(\mathcal{H})$ such that $B = AD$.

Moreover, in this case there exists a unique solution $D$ of the equation $AX = B$ such that $R(D) \subseteq \overline{R(A)}$. The operator $D$ is called the reduced solution of the equation $AX = B$ and $\|D\|^2 = \min\{\mu : BB^* \leq \mu AA^*\}$. If $R(A)$ is closed, then $D = A^\dagger B$.

**Corollary 2.2.** Let $A, B \in L(\mathcal{H})^+$. Then there exists $\mu > 0$ such that $A - \mu B \geq 0$ if and only if $R(B^{1/2}) \subseteq R(A^{1/2})$. In this case, if $B \neq 0$ and $D$ is the reduced solution of the equation $A^{1/2}X = B^{1/2}$, we have

$$\max \{ \mu \geq 0 : A - \mu B \geq 0 \} = \|D\|^{-2}.$$ 

If $R(A)$ is closed, this number coincides with $\rho(A^{1/2})^{-1} = \|B^{1/2}A^{1/2}\|^{-1}$.
Proof. If \( R(A) \) is closed, then \( R(A) = R(A^{1/2}) \) and \( D = (A^{1/2})^* B^{1/2} \). Hence
\[
\|D\|^2 = \|D^* D\| = \|B^{1/2} A^{1/2} B^{1/2}\| = \rho(B^{1/2} A^{1/2} B^{1/2}) = \rho(A^1 B).
\]
\[\]

**Corollary 2.3.** Let \( A \in L(\mathcal{H})^+ \) and \( \xi \in \mathcal{H} \) with \( \|\xi\| = 1 \). Consider the rank one projection \( P = \xi \otimes \xi = P_\xi \) onto the subspace generated by \( \xi \). If
\[
I_\xi(A) = \max \{ \mu \geq 0 : A - \mu P \geq 0 \},
\]
then \( I_\xi(A) \neq 0 \iff \xi \in R(A^{1/2}) \). In this case, if \( \eta \in \ker A^\perp \) satisfies \( A^{1/2} \eta = \xi \), we get \( I_\xi(A) = \|\eta\|^{-2} \). If \( \xi \in R(A) \), then for every \( \zeta \in \mathcal{H} \) such that \( A\zeta = \xi \) it holds
\[
I_\xi(A) = (A\zeta, \zeta)^{-1}.
\]
If \( R(A) \) is closed, then \( I_\xi(A) = (A^1 \xi, \xi)^{-1} \).

**Proof.** The first part follows from Corollary 2.2. Let \( \eta \in \ker A^\perp \) such that \( A^{1/2} \eta = \xi \). Then the reduced solution of the equation \( A^{1/2} X = P \) is \( \eta \otimes \xi \), the one rank operator defined by
\[
\eta \otimes \xi(\gamma) = \langle \gamma, \xi \rangle \eta, \quad \gamma \in \mathcal{H}.
\]
It is easy to see that \( \|\eta \otimes \xi\| = \|\xi\| \|\eta\| = \|\eta\| \). If there exists \( \zeta \in \mathcal{H} \) such that \( A\zeta = \xi \), then \( A^{1/2} \zeta = \eta \) and \( \langle A\zeta, \zeta \rangle = \|\eta\|^2 \). If \( R(A) \) is closed, then \( \eta = (A^{1/2})^\dagger \xi \), so that \( \|\eta\|^2 = \langle \eta, \eta \rangle = \langle A^1 \xi, \xi \rangle \).

2.4. Suppose that \( \dim \mathcal{H} = n < \infty \). We identify \( L(\mathcal{H}) \) with \( M_n(\mathbb{C}) \), the algebra of \( n \times n \) complex matrices. Let \( A \in L(\mathcal{H})^+ \). In [16] the notion of minimal index for \( A \) was defined as
\[
I(A) = \max \{ \mu \geq 0 : A \circ B \geq \mu \cdot B \quad \forall \ B \in L(\mathcal{H})^+ \}
= \max \{ \mu \geq 0 : \Phi_A - \mu \cdot \text{Id} \geq 0 \quad \text{on} \ L(\mathcal{H})^+ \}
= \max \{ \mu \geq 0 : A - \mu \cdot ee^t \geq 0 \}
\]
where \( e = (1, \ldots, 1) \), the symbol \( \circ \) denotes the Hadamard product of matrices and \( \Phi_A(C) = A \circ C, \ C \in L(\mathcal{H}) \). The last equality follows from the fact that for \( C \in L(\mathcal{H}) \), \( \Phi_C \geq 0 \iff C \geq 0 \) (see [13]).

Note that, if \( \xi = n^{-1/2} e \), then \( \|\xi\| = 1 \) and \( I(A) = n^{-1} I_\xi(A) \). By Corollary 2.3, \( I(A) > 0 \) if and only if \( e \) belongs to the range of \( A \). In [16] and [7] it is shown that, in this case, for any vector \( y \) such that \( A(y) = e \),
\[
I(A) = (Ay, y)^{-1} = (A^1 e, e)^{-1} = \min \{ \langle Az, z \rangle : \langle z, e \rangle = 1 \}.
\]
Note that the first two equalities are particular cases of Corollary 2.3.
3 The shorted operator for selfadjoint operators

Let $A = A^* \in L(H)$ and $P \in \mathcal{P}$. We first need a characterization of those pairs $(A, P)$ such that, for some $\mu \in \mathbb{R}$, it holds

$$A - \mu P \geq 0.$$  \hfill (6)

The solution of this problem is well known, see for example [14]. We shall give a brief survey of the characterization of pairs $(A, P)$ satisfying equation (6), for the sake of completeness.

Note that if $Px = 0$ then $\langle (A - \mu P)x, x \rangle = \langle Ax, x \rangle$. Thus, a necessary condition for $A$ and $P$ to satisfy condition (6) is that $(1 - P)A(1 - P) \geq 0$.

Definition 3.1. Let $A \in L(H)$ such that $A = A^*$ and let $P \in \mathcal{P}$. We shall say that $A$ is $P$-positive if $(1 - P)A(1 - P) \geq 0$.

Remark 3.2. Let $e = (1, \ldots, 1) \in \mathbb{C}^n$ and let $P_e \in M_n(\mathbb{C})$ denote the orthogonal projection onto the subspace generated by $e$. A real symmetric matrix $A \in M_n(\mathbb{R})$ is called almost positive if $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{R}^n$ such that $\langle \xi, e \rangle = 0$. Therefore a real selfadjoint matrix $A \in M_n(\mathbb{C})$ is almost positive if and only if it is $P_e$-positive.

Proposition 3.3. Let $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$ and $A \in L(H)$ be hermitian and $P$-positive. Let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be the representation induced by $P$. Then the following conditions are equivalent:

1. There exists $\mu \in \mathbb{R}$ such that $A - \mu P \geq 0$.

2. The partial matrix $\begin{pmatrix} a & b \\ b^* & ? \end{pmatrix}$ admits a positive completion.

3. The set $M(A, \mathcal{S}) = \{ D \in L(H) : D = D^*, D \leq A, R(D) \subseteq \mathcal{S} \}$ is not empty.

4. There exists $x \in L(\mathcal{S}, \mathcal{S}^\perp)$ such that $b = a^{1/2}x$.

5. $R(b) \subseteq R(a^{1/2})$.

and, if $R(a)$ is closed, also

6. $\ker a = \ker A \cap \mathcal{S}^\perp$.

Proof. 1 $\rightarrow$ 2: Take $d = c - \mu P$. Then $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = A - \mu P \geq 0$.

2 $\rightarrow$ 3: Let $d \in L(\mathcal{S})$ such that $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \geq 0$. Then $D = \begin{pmatrix} 0 & 0 \\ 0 & c - d \end{pmatrix} \in M(A, \mathcal{S})$.

3 $\rightarrow$ 1: If $D \in M(A, \mathcal{S})$, take $\mu \in \mathbb{R}$ such that $-D \leq -\mu P$.

4 $\leftrightarrow$ 5: It is a consequence of Douglas Theorem 2.1.
2 ↔ 5: It is well known (see [1] or [14]). For example, if \( b = a^{1/2}x \) with \( x \in L(S, S^\perp) \), then
\[
\begin{pmatrix}
a & b \\
b^* & x^*x
\end{pmatrix} = \begin{pmatrix}
a^{1/2} & 0 \\
x^* & 0
\end{pmatrix} \begin{pmatrix}
a^{1/2} & x \\
0 & 0
\end{pmatrix} \geq 0.
\]
If \( R(a) \) is closed, then \( R(a^{1/2}) = R(a) = (\ker a)^\perp \). In this case
\[
R(b) \subseteq R(a) \iff \ker a \subseteq \ker b^* \iff (\forall \xi \in S^\perp, a\xi = 0 \Rightarrow a\xi + b^*\xi = A\xi = 0),
\]
i.e. condition 5 is equivalent to \( \ker a \subseteq \ker A \cap S^\perp \). Note that the reverse inclusion always holds.

**Remark 3.4.** With the notations of Proposition 3.3, if \( R(a) \) is not closed, then conditions 1 – 5 still imply, with the same proof, that \( \ker a = \ker A \cap S^\perp \).

**Definition 3.5.** Let \( A \in L(H) \) be hermitian and \( P \in \mathcal{P} \) such that \( A \) is \( P \)-positive.

1. \( A \) is called \( P \)-complementable if any of the conditions of Proposition 3.3 holds.

2. In this case we define: \( I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\} \).

If \( A \) is \( P \)-complementable, the shorted operator can be defined for the pair \((A, P)\), and several results for shorted operators of positive operators (see Anderson Trapp [1]) remain true in this case. We show these properties in the rest of this section.

**Definition 3.6.** Let \( P \in \mathcal{P} \) with \( R(P) = S \) and let \( A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(H) \) be hermitian \( P \)-complementable. Let \( d \in L(S, S^\perp) \) be the reduced solution of the equation \( b = a^{1/2}x \). Then we define the Schur complement (or shorted operator) of \( A \) with respect to \( S \) as
\[
\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}.
\]

**Proposition 3.7.** Let \( P \in \mathcal{P} \) with \( R(P) = S \) and let \( A \) be \( P \)-complementable.

1. If \( A \geq 0 \), then \( \Sigma(A, P) \) is the usual shorted operator for \( A \) and \( S \).

2. Let \( \mu \in \mathbb{R} \). Then \( \Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P \).

3. \( \Sigma(A, P) = \max\{D \in L(H) : D = D^*, D \leq A, R(D) \subseteq S\} \).

4. \( \Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = S\} \).

5. Let \( \xi \in S \). Then
\[
\langle \Sigma(A, P)\xi, \xi \rangle = \inf\{\langle A(\xi + \eta), \xi + \eta \rangle, \eta \in S^\perp\}.
\]
6. If \( a = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \) and \( R(a) \) is closed, then

\[
\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - b^*a^\dagger b \end{pmatrix}.
\]

where \( a^\dagger \) is the Moore-Penrose pseudoinverse of \( a \) in \( L(S^\perp) \).

**Proof.**

1. It is shown in Anderson Trapp [1].

2. It is clear by definition.

3. If \( A \geq 0 \), then \( \Sigma(A, P) = \max\{D \in L(H) : D \geq 0, D \leq A \text{ and } R(D) \subseteq S\} \) (see [1]). The general case can be easily deduced from the positive case using item 2.

4. If \( A \geq 0 \), then \( \Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = S\} \) (see [1]). The general case can be easily deduced from the positive case using item 2 and fact that, if \( Q \in \mathcal{Q} \) has \( R(Q) = S \), then \( QPQ^* = P \).

5. The positive was shown in [1]. If \( A \not\geq 0 \), denote by \( B = A - I_P(A)P \geq 0 \). By item 2, \( \Sigma(A, P) = \Sigma(B, P) + I_P(A)P \). Thus,

\[
\langle \Sigma(B, P)\xi, \xi \rangle = \inf\{\langle B(\xi + \eta), \xi + \eta \rangle, \eta \in S^\perp\}
= \inf\{\langle A(\xi + \eta), \xi + \eta \rangle, \eta \in S^\perp\} - I_P(A)\|\xi\|^2.
\]

6. If \( R(a) \) is closed, then \( R(a^{1/2}) \) is also closed, \((a^{1/2})^\dagger = (a^\dagger)^{1/2} \) and \( d = (a^{1/2})^\dagger b \) is the reduced solution of the equation \( a^{1/2}x = b \).

**Remark 3.8.** The following properties are easy consequences of Proposition 3.7 and the corresponding results for the positive case (see [1] and Li-Mathias [11]):

1. Let \( P, Q \in \mathcal{P} \) such that \( P \leq Q \), let \( A \in L(H) \) \( P \)-complementable and \( B \in L(H) \) such that \( A \leq B \). Then \( B \) is \( P \)-complementable, \( \Sigma(A, P) \leq \Sigma(B, P) \), \( A \) is \( Q \)-complementable and \( \Sigma(A, P) \leq \Sigma(A, Q) \).

2. Let \( \{E_n\} \in L(H) \) be a monotone decreasing sequence of positive operators strongly convergent to 0 and let \( A \in L(H) \) be \( P \)-complementable. Then \( \Sigma(A + E_n, P) \) converges strongly to \( \Sigma(A, P) \).

3. Let \( A \in L(H) \) be an invertible \( P \)-complementable operator. Then

\[
\|\Sigma(A + \epsilon, P) - \Sigma(A, P)\| \to 0 \text{ as } \epsilon \to 0^+.
\]

4. Let \( A \in L(H) \) be \( P \)-complementable. Then there exist unique operators \( F \) and \( G \) such that \( A = F + G \) with \( R(F) \subseteq S \), \( G \geq 0 \) and \( R(G^{1/2}) \cap S = \{0\} \).
5. Let $A \in L(\mathcal{H})$ and $P \in \mathcal{P}$ such that $A$ is $P$-complementable. Let $f$ be an operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geq 0$. Then $\Sigma(f(A), P) \geq f(\Sigma(A, P))$.

6. Let $\{P_n\} \in \mathcal{P}$ be a decreasing sequence of projections such that $P_n \xrightarrow{S.O.T} P$ and let $A \in L(\mathcal{H})$ be $P$-complementable. Then $\{\Sigma(A, P_n)\}$ decreases to $\Sigma(A, P)$ (see [14] or [2]).

4. $A$-selfadjoint projections

Given $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$ and $A \in L(\mathcal{H})$ $P$-positive, we shall consider a condition stronger than being $P$-complementable which is the existence of $A$-selfadjoint projections onto $\mathcal{S}^\perp$, i.e., $Q \in \mathcal{Q}$ such that $AQ = Q^*A$ and $R(Q) = \mathcal{S}^\perp$.

Definition 4.1. Let $A = A^* \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. We denote by

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}^\perp, AQ = Q^*A\}.$$ 

The pair $(A, \mathcal{S})$ is said to be compatible if $\mathcal{P}(A, \mathcal{S})$ is not empty.

The notion of a compatible pair was introduced in [4], where a characterization of compatible pairs $(A, \mathcal{S})$ in terms of the Schur complements $\Sigma(A, P)$ is given, in case that $A \geq 0$. The following two results are taken from [4]:

Lemma 4.2. Let $A = A^* \in L(\mathcal{H})$ and $Q \in \mathcal{Q}$. Then the following conditions are equivalent:

1. $Q$ satisfies that $AQ = Q^*A$, i.e. $Q$ is $A$-selfadjoint.
2. $\ker Q \subseteq A^{-1}(R(Q)^\perp)$.

and, if $A \geq 0$,

3. $Q^*AQ \leq A$.

Proposition 4.3. Given $A = A^* \in L(\mathcal{H})$ and $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$, the following conditions are equivalent:

1. The pair $(A, \mathcal{S})$ is compatible (i.e. $\mathcal{P}(A, \mathcal{S})$ is not empty).

2. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ then $R(b) \subseteq R(a)$.

3. $\mathcal{S}^\perp + A^{-1}(\mathcal{S}) = \mathcal{H}$.

In this case, for every $E \in \mathcal{P}(A, \mathcal{S})$, $\ker E \subseteq A^{-1}(\mathcal{S})$. 

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Corollary 4.4. If $(A, S)$ is compatible and $A$ is $P$-positive, then $A$ is $P$-complementable.

Proof. Just note that, if $a = (1 - P)A(1 - P) \geq 0$, then $R(a) \subseteq R(a^{1/2})$. 

Remark 4.5. Let $A \in L(H)$ be hermitian and $P \in P$ with $R(P) = S$ such that $A$ is $P$-positive and suppose that $R((1 - P)A(1 - P))$ is closed. Then $(A, S)$ is compatible if and only if $A$ is $P$-complementable. This last condition holds whenever $\dim S^\perp < \infty$. Therefore if $H$ is a finite dimensional space and $A$ is $P$-positive, the conditions $(A, S)$ is compatible and $A$ is $P$-complementable are equivalent.

Proposition 4.6. Let $A = A^* \in L(H)$ such that $A$ is $P$-positive and the pair $(A, S)$ is compatible. Let $E \in \mathcal{P}(A, S)$ and $Q = I - E$. Then

1. $\Sigma(A, P) = AQ = Q^*A = Q^*AQ$.
2. $\Sigma(A, P) = \min\{FAF^* : F \in Q, R(F) = S\}$.
3. $R(\Sigma(A, P)) \subseteq R(A) \cap S$.

Proof. The case $A \geq 0$ was shown in [4] (with equality in item 3). The general case follows from the fact that if $F \in Q$ and $R(F) = S$, then $FP = PF^* = FPF^* = P$. Recall that if $B = A - I_P(A)P$, then $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$; and $R((I - E)^*) = \ker(I - E)^\perp = S$. Item 3 is clear because $R(AQ) \subseteq R(A)$.

Lemma 4.7. Let $A = A^* \in L(H)$ and $P \in P$ with $R(P) = S$. Suppose that $A$ is $P$-positive and $(A, S)$ is compatible. Let $E \in \mathcal{P}(A, S)$ and $Q = 1 - E$. Consider the operator $T = (1 - P) + Q$. Then

1. $T \in \text{Gl}(H)$ with $T^{-1} = E + P$.
2. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ in terms of $P$, then

$$T^*AT = \begin{pmatrix} a & 0 \\ 0 & \Sigma(A, P) \end{pmatrix}.$$  \hspace{1cm} (7)

3. If $A \in \text{Gl}(H)$ then $a \in \text{Gl}(S^\perp)$ and $\Sigma(A, P) \in \text{Gl}(S)$. Moreover, if we view $\Sigma(A, P) \in L(S)$, then $\Sigma(A, P)^{-1} = PA^{-1}P$ or, in other words,

$$\Sigma(A, P) = (PA^{-1}P)^\dagger.$$  \hspace{1cm} (8)

Proof.

1. Since $R(1 - P) = R(E) = \ker P = \ker Q = S^\perp$, then $(1 - P)E = E$ and $QP = Q$. Thus $T(E + P) = E + Q = 1$. The other case is similar.
2. The fact that $R(Q) = \ker E \subseteq A^{-1}(S)$ implies that $Q^*A(1 - P) = (1 - P)AQ = 0$. By Proposition 4.6, $Q^*AQ = \Sigma(A, P)$.

3. Note that $(T^*AT)^{-1} = T^{-1}A^{-1}(T^*)^{-1} = (E + P)A^{-1}(E^* + P)$. But $PE = E^*P = 0$, so that $\Sigma(A, P)^{-1} = P(T^*AT)^{-1}P = PA^{-1}P$.

Proposition 4.8. Let $A = A^* \in L(\mathcal{H})$ and $P \in \mathfrak{p}$ with $R(P) = S$. Suppose that $A$ is $P$-positive and $(A, S)$ is compatible. Then $R(\Sigma(A, P)) = R(A) \cap S$.

Proof. We use the notations of Lemma 4.7. By formula (7), $R(T^*AT) \cap S = R(\Sigma(A, P))$. On the other hand, if $\xi \in S$, then $T^*\xi = Q^*\xi = \xi$, because $R(Q^*) = \ker Q^\perp = S$ and $Q^* \in Q$. Hence $R(A) \cap S = R(\Sigma(A, P)) \cap S \subseteq R(T^*AT) \cap S = R(\Sigma(A, P))$. The reverse inclusion was shown in Proposition 4.6.

5 Computation of $I_P(A)$

Let $P \in \mathfrak{p}$ and $A = A^* \in L(\mathcal{H})$. Recall that, if $A$ is $P$-complementable, we have defined

$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}.$$  

Remark 5.1. If $A \geq 0$ then, by Corollary 2.2, $I_P(A) \neq 0$ if and only if $R(P) \subseteq R(A^{1/2})$ and, in this case, $I_P(A) = \|D\|^{-2}$, where $D$ is the reduced solution of the equation $A^{1/2}X = P$. Thus, if $R(A)$ is closed, then $I_P(A) = \rho(A^{1/2})$.

Suppose now that $A \not\geq 0$. It is easy to see that if $B = A + \mu P$, then $I_P(B) = I_P(A) + \mu$. Therefore a way to compute $I_P(A)$ would be to find a lower bound $\mu \leq I_P(A)$ in order to compute firstly $I_P(B)$ for $B = A - \mu P \geq 0$, reducing the general case to the positive case. Nevertheless this way seems to be not applicable. For example, it is easy to get, for any $M > 0$, selfadjoint matrices $A \in M_2(\mathbb{C})$ with $\|A\| \leq 2$ such that $I_P(A) < -M$, where $P$ is a fixed projection of rank one. Indeed, take $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix}$, for $\varepsilon < M^{-1}$.

We first show the key relation between $I_P(A)$ and the shorted operator $\Sigma(A, P)$:

Proposition 5.2. Let $A \in L(\mathcal{H})$ be hermitian, $A \not\geq 0$, and $P \in \mathfrak{p}$ with $R(P) = S$ such that $A$ is $P$-complementable. Then

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \min\{ \langle \Sigma(A, P)\xi, \xi \rangle : \xi \in S, \|\xi\| = 1 \}.$$  \hspace{1cm} (9)

Proof. Denote by $\mu = \lambda_{\min}(\Sigma(A, P))$. Since $A \not\geq 0$, it is easy to see that $\mu < 0$. In particular this shows the last equality in equation (9). Note that $\mu P \leq \Sigma(A, P)$, so that

$$A - \mu P \geq A - \Sigma(A, P) \geq 0,$$

and $\mu \leq I_P(A)$.

On the other hand, since $A - I_P(A)P \geq 0$, then $I_P(A)P \in M(A, S)$ and $I_P(A)P \leq \Sigma(A, P)$ (see Propositions 3.3 and 3.7), which implies that $I_P(A) \leq \mu$. 

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Remark 5.3. With the notations of Proposition 5.2, if \( A \geq 0 \), then the identity \( I_P(A) = \min\{ \langle \Sigma(A,P)\xi,\xi \rangle : \xi \in \mathcal{S}, \|\xi\| = 1 \} \) remains true; and this number coincides with \( \lambda_{\min}(\Sigma(A,P)) \) if we consider the spectrum of \( \Sigma(A,P) \) as an operator of \( L(\mathcal{S}) \) (in order to remove the number 0 if necessary).

The following properties of \( I_P(A) \) follow immediately from Remark 3.8 and Proposition 5.2.

**Corollary 5.4.** Let \( A \in L(\mathcal{H}) \) be hermitian and \( P \in \mathcal{P} \) such that \( A \) is \( P \)-complementable:

1. Let \( Q \in \mathcal{P} \) such that \( P \leq Q \) and suppose that \( A \not\geq 0 \). Then \( I_P(A) \leq I_Q(A) \). If \( A \geq 0 \) this property may fail because of the fact observed in Remark 5.3.

2. Let \( B \in L(\mathcal{H}) \) such that \( A \leq B \). Then \( I_P(A) \leq I_P(B) \).

3. Let \( \{E_n\} \in L(\mathcal{H}) \) be a monotone (not necessary strictly) decreasing sequence of positive operators strongly convergent to 0. Then the sequence \( \{I_P(A+E_n)\} \) decreases to \( I_P(A) \).

4. Let \( \{A_n\} \in L(\mathcal{H}) \) be a sequence of \( P \)-complementable operators which is norm convergent to an invertible \( P \)-complementable operator \( A \). Then \( \{I_P(A_n)\} \) converges to \( I_P(A) \).

5. Let \( f \) be a operator monotone map defined on \( \sigma(A) \cup \sigma(\Sigma(A,P)) \) such that \( f(0) \geq 0 \). Then \( I_P(f(A)) \geq f(I_P(A)) \).

6. Let \( \{P_n\} \) be a decreasing sequence of orthogonal projections such that \( P_n \xrightarrow{S.O.T} P \). Then \( \{I_{P_n}(A)\} \) decreases to \( I_P(A) \).

**Remark 5.5.** It was pointed out in [16] that the hypothesis in item 3 can not be relaxed, i.e the map \( A \mapsto I_P(A) \) is not norm continuous in general, as we see in the following example:

**Example 5.6.** Let \( a \neq 1 \) and \( \{b_n\} \subseteq \mathbb{R}_{>0} \) such that \( \lim_{n \to \infty} b_n = a \). Then the sequence of positive matrices

\[
A_n = \begin{pmatrix}
a^2 + a^{-2} & ab_n + (ab_n)^{-1} \\
ab_n + (ab_n)^{-1} & b_n^2 + b_n^{-2}
\end{pmatrix}
\]

converges in norm to \( A = (a^2 + a^{-2})ee^\dagger \), where \( e = (1,1) \). Let \( x_n = (a,b_n) \) and \( y_n = (a^{-1},b_n^{-1}) \). Note that \( A_n = x_n^*x_n + y_n^*y_n \) and \( e = \lambda_nx_n + \mu_ny_n \), with \( \lambda_n = (a+b_n)^{-1} \) and \( \mu_n = ab_n(a+b_n)^{-1} \). If a vector \( z \) satisfies that \( A_nz = e \), then

\[
e = A_nz = (x_n^*x_n + y_n^*y_n)z = \langle z, x_n \rangle x_n + \langle z, y_n \rangle y_n.
\]
\[ (z,e)^{-1} = (\langle z, x_n \rangle^2 + \langle z, y_n \rangle^2)^{-1} = \frac{(a+b_n)^2}{1+a^2b_n^2}. \] Since \( I_{P_n}(A_n) = 2\langle A_n^{-1}e, e \rangle^{-1} = \frac{2(a+b_n)^2}{1+a^2b_n^2} \), we get

\[ \lim_{n \to \infty} I_{P_n}(A_n) = \frac{8}{a^2 + a^{-2}} \neq 2(a^2 + a^{-2}) = I_{P_n}(A) \]

The following results are the natural generalizations of formula (5) to our setting:

**Corollary 5.7.** Let \( A \in \mathcal{L}(\mathcal{H}) \) be hermitian and \( P \in \mathcal{P} \) with \( R(P) = \mathcal{S} \) such that \( A \) is \( P \)-complementable. Then

\[ I_{P}(A) = \inf \{ \langle A\xi, \xi \rangle : \xi \in \mathcal{H}, \| P\xi \| = 1 \} \quad (10) \]

**Proof.** It is a consequence of equation (9) in Proposition 5.2 (or Remark 5.3 in case that \( A \geq 0 \)) and item 5 of Proposition 3.7. \( \blacksquare \)

**Corollary 5.8.** Let \( A \) and \( P \) be as above and suppose that \( P = \xi \otimes \xi \) for some unit vector \( \xi \in \mathcal{H} \). Then

\[ I_{P}(A) = \inf \{ \langle A\eta, \eta \rangle : \eta \in \mathcal{H}, \langle \eta, \xi \rangle = 1 \} \quad (11) \]

**Proof.** Note that \( P\eta = \langle \eta, \xi \rangle \xi \) and \( \| P\eta \| = |\langle \eta, \xi \rangle| \). Also, if \( \omega \in \mathbb{C} \) has \( |\omega| = 1 \), then \( \langle A\omega\eta, \omega\eta \rangle = \langle A\eta, \eta \rangle \). \( \blacksquare \)

Throughout, we shall consider \( P \in \mathcal{P} \) with \( R(P) = \mathcal{S} \) and \( A \in \mathcal{L}(\mathcal{H}) \) \( P \)-positive such that \((A, \mathcal{S})\) is compatible. In this case almost all results which can be shown for matrices can be extended to the infinite dimensional case.

**Remark 5.9.** Let \( A \in \mathcal{L}(\mathcal{H}) \) be hermitian and \( P \in \mathcal{P} \) with \( R(P) = \mathcal{S} \) such that \((A, \mathcal{S})\) is compatible. Suppose that \( I_{P}(A) \neq 0 \). Then

\[ R(A) \cap \mathcal{S} = R(\Sigma(A,P)) \neq \{0\}. \]

Indeed, since \((A, \mathcal{S})\) is compatible, \( R(\Sigma(A,P)) = R(A) \cap \mathcal{S} \) by Proposition 4.8. On the other hand, \( 0 \neq I_{P}(A) = \lambda_{\min}(\Sigma(A,P)) \), by Proposition 5.2. Hence \( \Sigma(A,P) \neq 0 \).

**Theorem 5.10.** Let \( A \in \mathcal{L}(\mathcal{H}) \) be hermitian and \( P \in \mathcal{P} \) with \( R(P) = \mathcal{S} \), such that \( A \) is \( P \)-positive and \((A, \mathcal{S})\) is compatible with \( I_{P}(A) \neq 0 \). Denote by \( \mathcal{T} = \mathcal{S} \cap \overline{R(A)} \) and \( Q = P \mathcal{T} \). Then

1. \( A \) is \( Q \)-complementable. Moreover, the pair \((A, \mathcal{T})\) is compatible.

2. \( \Sigma(A, P) = \Sigma(A, Q) \).

3. \( I_{P}(A) = I_{Q}(A) \).
Proof. If \( A \geq 0 \), by Remark 5.3, we know that \( \Sigma(A, P) \) is invertible in \( L(S) \). On the other hand, since \((A, S)\) is compatible, \( S = R(\Sigma(A, P)) = R(A) \cap S \subseteq R(A) \).

Suppose now that \( A \notin 0 \). By Remark 5.9, \( R(\Sigma(A, P)) = R(A) \cap S \subseteq T \). Hence

\[
\Sigma(A, P) \in M(A, T) = \{ D \in L(\mathcal{H}) : D = D^*, D \leq A, R(D) \subseteq T \} \neq \emptyset.
\]

Therefore, by Proposition 3.3, \( A \) is \( Q \)-complementable and, by Proposition 3.7, \( \Sigma(A, P) \leq \Sigma(A, Q) \). The inequality \( \Sigma(A, Q) \leq \Sigma(A, P) \) follows by Remark 3.8. Then,

\[
I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \lambda_{\min}(\Sigma(A, Q)) = I_Q(A).
\]

Using Proposition 4.3 item 3, in order to show that the pair \((A, T)\) is compatible, it suffices to verify that \( T^+ + A^{-1}(T) = \mathcal{H} \), which follows from the following facts:

\( S^+ + A^{-1}(S) = \mathcal{H} \) (since \((A, S)\) is compatible), \( S^+ \subseteq T^+ \) and \( A^{-1}(S) = A^{-1}(S \cap R(A)) \subseteq A^{-1}(T) \).

\[ \blacksquare \]

Remark 5.11. When \( \dim S = 1 \), if \( A \) is \( P \)-positive and \( P \)-compatible we can deduce that \( S \subseteq R(A) \). More generally, if \( \dim S < \infty \), \( A \) is injective and \((A, S)\) is compatible, then \( S \subseteq R(A) \). Indeed, note that \( \dim A^{-1}(S) = \dim S \cap R(A) \), and \( A^{-1}(S) \) must be a supplement of \( S^+ \). Nevertheless, if we remove the condition \((A, S)\) is compatible, this is not true, even if \( \dim S = 1 \) and \( A \) is injective and \( P \)-complementable, as the following example shows:

Example 5.12. Let \( A \in L(\mathcal{H})^+ \) be injective non-invertible. Let \( \xi \in \mathcal{H} \setminus R(A) \) be a unit vector. Denote by \( S \) the subspace generated by \( \xi \), \( P = P_S \). If

\[
A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}
\]

in terms of \( P \) and \( A\xi = \lambda\xi + \eta \) with \( \eta \in S^+ \), then \( \lambda = \langle A\xi, \xi \rangle \neq 0 \) and \( \eta \neq 0 \) (otherwise \( \xi \in R(A) \)). Therefore \( c = \lambda P \) and \( b(\mu\xi) = \mu\eta, \mu \in \mathbb{C} \).

Suppose that \( \eta \in R(a) \), i.e. there exists \( \nu \in S^+ \) which verifies \( \nu \eta = b\xi \). Then

\[
(1 - P)A(\nu - \xi) = a\nu - b\xi = 0, \text{ so } A(\nu - \xi) \text{ is a multiple of } \xi, \text{ which must be } 0 \text{ (} \xi \notin R(A) \text{). So } \nu = \xi, \text{ a contradiction. Therefore } R(b) \nsubseteq R(a) \text{ and the pair } (A, S) \text{ is incompatible.}
\]

Now consider \( B = A + \mu P \), for any \( \mu \in \mathbb{R} \). It is clear that \( B \) must be \( P \)-complementable \( (B - \mu P = A \geq 0) \). But the facts that \( A \) is injective and \( \xi \notin R(A) \), clearly imply that \( B \) is injective and \( \xi \notin R(B) \).

5.13. Fix \( E \in \mathbb{F} \) with range \( \mathcal{M} \). Denote by \( L(\mathcal{H})\mathcal{M} = \{ C \in L(\mathcal{H}) : ECE = C \} \). For \( C \in L(\mathcal{H})\mathcal{M} \), denote by \( C_0 \in L(\mathcal{M}) \) the compression of \( C \) to \( \mathcal{M} \). With respect to the matrix representation induced by \( E \)

\[
C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix} \mathcal{M}^+ \mathcal{M}.
\]

The following properties of this compression are easy to see:
1. The map \( L(\mathcal{H})_M \ni C \mapsto C_0 \in L(\mathcal{M}) \) is a \(*\)-isomorphism of \( C^*\)-algebras, i.e. it is isometric and compatible with sums, products and adjoints.

2. If \( C = C^* \in L(\mathcal{H})_M \) and \( R(C) = \mathcal{M} \), then \( C_0 \in Gl(\mathcal{M}) \) and \( (C_0)^{-1} = (C^\dagger)_0 \). If \( R(C) \) is closed, then \( (C^\dagger)_0 = (C^\dagger) \).

**Theorem 5.14.** Let \( A \in L(\mathcal{H}) \) be hermitian, \( A \not\succeq 0 \), and \( P \in \mathcal{P} \) with \( R(P) = \mathcal{S} \) such that \( A \) is \( P \)-positive and \( (A, \mathcal{S}) \) is compatible. Suppose that \( R(A) \) is closed. Denote by \( T = \mathcal{S} \cap R(A) \) and \( Q = P_T \). Then

\[
I_P(A) = I_Q(A) = \lambda_{\min}(QA\dagger Q)^\dagger.
\]  

**Proof.** Since we only need to prove the equality \( I_Q(A) = \lambda_{\min}(QA\dagger Q)^\dagger \), we shall directly suppose that \( R(P) \subseteq R(A) \). Denote \( \mathcal{M} = R(A) \) and \( E = P_M \). Using the notations of 5.13, we have that \( \Sigma(A, P)_0 = \Sigma(A_0, P_0) \), \( I_P(A) = I_{P_0}(A_0) \) and \( A_0 \) is invertible. Therefore, by Lemma 4.7,

\[
\Sigma(A_0, P_0) = (P_0(A_0)^{-1}P_0)^\dagger = (PA\dagger P)_0^\dagger = ((PA\dagger P)^\dagger)_0
\]

and

\[
I_P(A) = I_{P_0}(A_0) = \lambda_{\min}\Sigma(A_0, P_0) = \lambda_{\min}((PA\dagger P)^\dagger)_0 = \lambda_{\min}(PA\dagger P)^\dagger.
\]

\[\Box\]

### 6 Some applications

The problem of calculating \( I_P(A) \) of a \( P \)-complementable operator \( A \) with respect to a projection \( P \) has already been considered for certain projections \( P \), mainly in the finite dimensional case. R. Reams [15] showed that if \( A \in M_n(\mathbb{R}) \) is invertible and almost positive (see Remark 3.2), then \( A \) is \( P_e \)-complementable and \( I_{P_e}(A) = n \cdot \langle A^{-1}e, e \rangle^{-1} \), where \( e = (1, \ldots, 1) \in \mathbb{C}^n \) and \( P_e \) denotes the orthogonal projection onto the subspace generated by \( e \). We obtain a generalization of this result in the non-positive case. The general positive case was already considered in [7] and Corollary 2.3 (for every unit vector \( \xi \in \mathbb{C}^n \)).

**Corollary 6.1.** Let \( \xi \in \mathbb{C}^n \) be a unit vector. Let \( A \in M_n(\mathbb{C}) \) be non-positive but \( P_\xi \)-positive. Then \( A \) is \( P_\xi \)-complementable if and only if

\[
\forall \eta \in \mathbb{C}^n, \quad \langle \eta, \xi \rangle = 0 \quad \text{and} \quad \langle A\eta, \eta \rangle = 0 \Rightarrow A\eta = 0 \quad (13)
\]

In this case \( \xi \in R(A) \) and

\[
I_{P_\xi}(A) = \langle A^\dagger \xi, \xi \rangle^{-1} = \min\{\langle Az, z \rangle : \langle z, \xi \rangle = 1\} \quad (14)
\]
Proof. Condition (13) is equivalent to \( \ker((1 - P_\xi)A(1 - P_\xi)) \cap \{ \xi \}^\perp = \ker(A) \cap \{ \xi \}^\perp \). By Proposition 3.3, this is equivalent to the fact that \( A \) is \( P_\xi \)-complementable, since \( R(A) \) is closed. Note that \( I_{P_\xi}(A) < 0 \), since \( A \not\geq 0 \). By Remarks 4.5 and 5.9 we get \( R(A) \cap R(P_\xi) \neq \{0\} \). Therefore \( \xi \in R(A) \) and \( \langle A^\dagger \xi, \xi \rangle \neq 0 \). By equation (12) in Theorem 5.14,

\[
I_{P_\xi}(A) = \lambda_{\min}(P_\xi A^\dagger P_\xi)^\dagger = \lambda_{\min}(\langle A^\dagger \xi, \xi \rangle P_\xi)^\dagger = \langle A^\dagger \xi, \xi \rangle^{-1}.
\]

In order to prove equation (14), it only remains to show that the infimum in equation (11) is actually a minimum. Let \( \zeta = A^\dagger \xi \) and \( \eta = \langle A^\dagger \xi, \xi \rangle^{-1} \zeta \). Then

\[
\langle A\eta, \eta \rangle = \langle A^\dagger \xi, \xi \rangle^{-2} \langle A\zeta, \zeta \rangle = \langle A^\dagger \xi, \xi \rangle^{-2} \langle A^\dagger \xi, \xi \rangle = \langle A^\dagger \xi, \xi \rangle^{-1},
\]

and the minimum is attained at \( \eta \). It was also noted in [15] that the problem of calculating \( I_P(A) \) with respect to \( P = P_\xi \) is equivalent to a problem posed by Fiedler and Markham in [9], that is to calculate

\[
\max \{ \lambda_{\min}((A \circ C)C^{-1}), C > 0 \}
\]

for a positive matrix \( A \in M_n(\mathbb{C}) \), where \( A \circ B \) denotes the Hadamard product of \( A \) and \( B \). The corollary above complements the results obtained in [9] in the non-positive, non-invertible case.

Recall that given a positive matrix \( A \in M_n(\mathbb{C}) \), the minimal index was introduced in [16] as

\[
I_A = \max \{ \mu \geq 0 : A \circ B \geq \mu B, \ B \geq 0 \}.
\]

Given \( P \in M_n(\mathbb{C}) \) an orthogonal projection and a \( P \)-complementable matrix \( A \), there is a relation between \( I_P(A) \) and the Schur multiplier induced by \( A \):

**Corollary 6.2.** Let \( M = \{ x_1, ..., x_k \} \subseteq \mathbb{C}^n \) be an orthonormal set and let \( P \) be the orthogonal projection onto the subspace spanned by \( M \). Suppose that \( A \in M_n(\mathbb{C}) \) is \( P \)-complementable. Then

\[
I_P(A) = \max \{ \mu \in \mathbb{R} : A \circ B \geq \mu \sum_{i=1}^k D_x BD_{x_i}^*, \ B \geq 0 \} \tag{15}
\]

where \( D_x \) denotes the diagonal matrix with main diagonal \( x \in \mathbb{C}^n \).

**Proof.** First note that \( P = \sum_{i=1}^k x_i x_i^* \). Thus \( A - \mu P \geq 0 \) if and only if every \( B \geq 0 \) satisfies \( (A - \mu \sum_{i=1}^k x_i x_i^*) \circ B \geq 0 \), which is equivalent to \( A \circ B \geq \mu \sum_{i=1}^k D_x BD_{x_i}^* \), since a simple calculation shows that \( C \circ xx^* = D_x CD_x^* \) for every \( C \in M_n(\mathbb{C}) \) and \( x \in \mathbb{C}^n \). This shows formula (15).
Completely positive maps on $M_n(\mathbb{C})$

**Definition 6.3.** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. $\Phi$ is positive if $\Phi(A) \succeq 0$ whenever $A \succeq 0$. $\Phi$ is selfadjoint if $\Phi(A^*) = \Phi(A)^*$ or equivalently if $\Phi(A)$ is selfadjoint whenever $A$ is selfadjoint.

Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. If $m \in N$, we denote $\Phi^{(m)} : M_m(M_n(\mathbb{C})) \to M_m(M_n(\mathbb{C}))$ the map given by

$$\Phi^{(m)}((a_{ij})) = (\Phi(a_{ij}))_{ij}, \quad (a_{ij}) \in M_m(M_n(\mathbb{C})), \quad$$

and call it the inflation of order $m$ of $\Phi$.

**Definition 6.4.** The linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called completely positive if $\Phi^{(m)}$ is positive for every $m \in N$.

In the following, $\{e_{ij}\} \subseteq M_n(\mathbb{C})$ denotes the canonical basis for $M_n(\mathbb{C})$. Now we state a result due to M.D. Choi ([3]):

**Theorem 6.5.** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then $\Phi$ is completely positive if and only if $\Phi^{(m)}((e_{ij})) = (\Phi(e_{ij}))_{ij} \in M_n(M_n(\mathbb{C}))$ is positive.

**Remark 6.6.** Note that the matrix $E = ((e_{ij}))_{ij} \in M_n(M_n(\mathbb{C})) \simeq M_{n^2}(\mathbb{C})$ is a scalar multiple of a rank one projection. Indeed, if $\{e_i\}$ denotes the canonical basis of $\mathbb{C}^n$ and $v \in \mathbb{C}^{n^2}$ is the vector $v = (e_1, \ldots, e_n)$, then $(e_{ij})_{ij} = vv^*$. Thus $E = \frac{1}{n}P_v$, where $P_v$ is the projection onto the subspace generated by $v$.

**Remark 6.7.** Let $A \in M_n(A)$. Then the linear map $\Phi_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $\Phi_A(B) = A \circ B$ is selfadjoint (resp. positive) if and only if $A$ is selfadjoint (resp. positive). Moreover, if $A \succeq 0$, then $\Phi_A$ is completely positive, since the inflated matrix $A^{(n)} \succeq 0$ and $\Phi_A^{(n)} = \Phi_{A^{(n)}}$ (see [13]). Therefore $\Phi_A - \mu \text{Id}$ is completely positive if and only if $A - \mu ee^* \succeq 0$, where $e \in \mathbb{C}^n$ is given by $e = (1, \ldots, 1)$. Note that $ee^* = n P_e$, since $\|e\| = n^{1/2}$. Therefore we conclude that for every $P_e$-complementable matrix $A$,

$$I(A) = \max\{\mu \in \mathbb{R} : \Phi_A - \mu \text{Id is completely positive}\} = \frac{1}{n} I_{P_e}(A),$$

where $I(A)$ is the minimal index of $A$ defined in 2.4 (in fact, its natural generalization for $A$ not necessarily positive, but $P_e$-complementable).

**Definition 6.8.** Let $\Phi : M_n(C) \to M_n(C)$ be a selfadjoint map. We say that $\Phi$ is complementable if there exists $\mu \in \mathbb{R}$ such that $\Phi - \mu \text{Id}$ is completely positive. In this case we define:

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu \text{Id is completely positive}\}.$$
Note that all completely positive maps $\Phi$ are complementable and $I(\Phi) \geq 0$. But in general not all selfadjoint maps are complementable. For example, if $A \in M_n(\mathbb{C})$ is selfadjoint, then $\Phi_A$ is complementable if and only if $A$ is $P_v$-complementable.

**Theorem 6.9.** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a selfadjoint map. Then, with the notations of Remark 6.6,

1. Suppose that $\Phi$ is not completely positive. In this case $\Phi$ is complementable if and only if for all $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$

$$\sum_{i=1}^n (\eta_i)_i = 0 \Rightarrow \sum_{i,j=1}^n \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle \geq 0 \quad (16)$$

and

$$\sum_{i=1}^n (\eta_i)_i = 0 \quad \text{and} \quad \sum_{i,j=1}^n \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle = 0 \Rightarrow \sum_{j=1}^n \Phi(e_{ij}) \eta_j = 0, \ i = 1, \ldots, n \quad (17)$$

or, equivalently, if $A_\Phi = \Phi^{(n)} E = (\Phi(e_{ij}))_{ij} \in M_{n^2}(\mathbb{C})$ is $P_v$-complementable.

2. In this case $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and we have

$$I(\Phi) = \min \{ \sum_{i,j=1}^n \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle : \eta_1, \ldots, \eta_n \in \mathbb{C}^n \text{ and } \sum_{i=1}^n (\eta_i)_i = 1 \}. \quad (18)$$

3. If conditions (16) and (17) hold, there exist $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$ such that

$$\sum_{j=1}^n \Phi(e_{ij}) \eta_j = e_i, \ i = 1, \ldots, n \quad (19)$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. For any such vectors,

$$I(\Phi)^{-1} = \sum_{i,j=1}^n \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle. \quad (20)$$

4. If $\Phi$ is completely positive then it is complementable and $I(\Phi) \geq 0$. Moreover, $I(\Phi) > 0$ if and only if there exist $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$ such that equation (19) holds. For any such vectors, equation (20) holds. Also equation (18) is true in this case.

**Proof.** From theorem 6.5 we conclude that the map $\Phi$ is complementable if and only if the matrix $A_\Phi = (\Phi(e_{ij}))_{ij}$ is $P_v$ complementable. It is easy to see that in fact $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$. Thus we can apply Corollary 6.1 to the matrix $A_\Phi \in M_{n^2}(\mathbb{C})$ and the projection $P_v$. Note that equation (16) holds if and only if $A_\Phi$ is $P_v$-positive and
condition (13) is equivalent to condition (17). Indeed, if \( \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n \) with \( \eta_i \in \mathbb{C} \) \((i = 1, \ldots, n)\), then \( \langle \eta, v \rangle = \sum_{i=1}^{n} \langle \eta_i \rangle \) and \( \langle A_\Phi \eta, \eta \rangle = \sum_{i,j=1}^{n} \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle \).

Note that condition (19) is equivalent to the fact that \( v \in R(A_\Phi) \), so this condition and equation (18) follow from equation (14). Similarly, \( I(\Phi) = n \cdot I_{P_v}(A_\Phi) = \langle v, A_\Phi \rangle \). Let \( \zeta = (\zeta_1, \ldots, \zeta_n) = A_\Phi^\dagger v \). If \( \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n \) satisfy condition (19) (i.e. \( A_\Phi \eta = v \)), then \( P_{R(A_\Phi)} \eta = \zeta \). Therefore

\[
\sum_{i,j=1}^{n} \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle = \langle A_\Phi \eta, \eta \rangle = \langle A_\Phi \zeta, \zeta \rangle = \langle v, A_\Phi \rangle = I(\Phi) - 1.
\]

Suppose now that \( \Phi \) is completely positive. It is clear that \( \Phi \) is complementable. By Corollary 2.3, it follows that \( I(\Phi) = n \cdot I_{P_v}(A_\Phi) > 0 \) if and only if \( v \in R(A_\Phi) \), since \( R(A_\Phi) \) is closed. This is equivalent to condition (19), and using Corollary 2.3, we can also deduce equations (20) and (18) in this case.

**Example 6.10.** Consider the map \( T : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) given by

\[
T(A) = \frac{1}{n} \text{Tr}(A) I_n = \frac{1}{n} \sum_{i,j=1}^{n} e_{ij}^* A e_{ij},
\]

where \( \text{Tr}(A) = \sum A_{ii} \) is the usual trace. Then \( T \) is completely positive; moreover it is a conditional expectation. Note that the matrix

\[
A_T = (T(e_{ij}))_{ij} = \frac{1}{n} I_{n^2}.
\]

Then \( I(T) > 0 \), since \( A_T(n e_i) = e_i \), \( 1 \leq i \leq n \), and \( T \) satisfies condition (19). Therefore, since \( T(e_{ij}) = 0 \) if \( i \neq j \) and \( T(e_{ii}) = \frac{1}{n} I_n \), using equation (20),

\[
I(T)^{-1} = \sum_{i=1}^{n} \langle T(e_{ii}) ne_i, ne_i \rangle = \sum_{i=1}^{n} n = n^2.
\]

This result is actually known in index theory of conditional expectations (using that \( T^{(n)}(P_v) = n^{-1} A_T = n^{-2} I_{n^2} \), see [12]). Note that the number

\[
J(T) = \max \{ \lambda \in \mathbb{R} : T - \lambda I \text{ is positive (not completely)} \} = n^{-1} \neq n^{-2} = I(T).
\]

Indeed, it is easy to see that \( A \geq 0 \) implies that \( \text{Tr}(A) \geq \rho(A) = \|A\| \), so that

\[
T(A) = \frac{1}{n} \text{Tr}(A) I_n \geq \frac{1}{n} A.
\]

Taking \( A = e_{11} \) we get \( T(A) \geq \lambda A \) if \( \lambda > \frac{1}{n} \); so that \( J(T) = n^{-1} \).
Remark 6.11. Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a selfadjoint map. The formulation of Theorem 6.9 intends to characterize complementability and to compute $I(\Phi)$ in terms of $\Phi$ itself instead of doing it in terms of the "inflated" matrix $A_{\Phi}$. Another way would be to recall the identity $I(\Phi) = n \cdot I_P(A_{\Phi})$ and use all the previous results of the paper. For example, let $U_1, \ldots, U_m \in M_n(\mathbb{C})$, and suppose that $\Phi$ is given by

$$\Phi(A) = \sum_{k=1}^m U_k^* A U_k, \quad A \in M_n(\mathbb{C}),$$

a prototypical completely positive map (see [3]). Denote by $V_k \in M_{n^2}(\mathbb{C})$ the block diagonal matrix with copies of $U_k$ in its diagonal. Denote by $v = (e_1, \ldots, e_n) \in \mathbb{C}^n$ and $E = (e_{ij})_{ij} = vv^*$. Note that $\|V_k v\| = \|U_k\|_2$ and $V_k^* E V_k = (V_k v)(V_k v)^*$. Therefore

$$A_{\Phi} = (\Phi(e_{ij}))_{ij} = \sum_{k=1}^m V_k^* E V_k = \sum_{k=1}^m \|U_k\|_2^2 P_{V_k v}.$$  

Thus $I(\Phi)$ can be computed using this expression and Corollaries 6.1 and 2.3.

References


