Finite element approximation of Maxwell eigenproblems on curved Lipschitz polyhedral domains

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Abstract

This paper deals with the finite element approximation of the spectral problem for the Maxwell equation on a curved non convex Lipschitz polyhedral domain Ω. Convergence and optimal order error estimates are proved for the lowest order edge finite element space of Nédelec on a tetrahedral mesh of approximate domains Ωh ⊂ Ω. These convergence results are based on the discrete compactness property which is proved to hold true also in this case.

Key words: Maxwell eigenvalue problem; curved domains; finite element methods; edge elements; discrete compactness property

1 Introduction

The approximation of the eigenvalue problem associated with Maxwell’s system is one of the most relevant problem in numerical simulation of electromagnetic phenomena.

Compared with other elliptic eigenvalue problems, Maxwell eigenproblem has some additional difficulties:

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- the commonly used variational formulations have a non empty essential spectrum; in fact, \( \sigma^{\text{ess}} = \{0\} \) with associated eigenspace being infinite dimensional,
- the associated energy space is not compactly embedded in \( L^2 \),
- the solutions exhibit a low regularity on non smooth domains; roughly speaking, are one order less regular than solutions of the corresponding Dirichlet problem.

We refer the reader to the book [34] or to the review paper [31] and the references therein. For a thorough discussion about singular solutions of time harmonic Maxwell equations, see [19].

These well known features have important consequences for the construction of the numerical approximation of the solution. In particular, a suitable numerical approximation should take care of the zero frequency infinite dimensional eigenspace. Otherwise, spurious modes may appear. This is the case, for instance, when nodal elements are used for represent each field component. Such spurious modes are eigenvalues of the discrete problem which have no physical meaning.

Several finite element methods have been developed to avoid this severe drawback; see, for instance, [10], [9], [17], [26], [4], [22], [38].

In recent years, a mathematical theory for conforming approximations have been completed. We refer the reader to the pioneering work [11] and to the fundamental work [16], where a set of sufficient and necessary conditions for a conforming finite element method to provide a spurious free approximation is given.

Even more recently, a theoretical framework for the analysis of the discontinuous Galerkin approximations of the Maxwell eigenproblem have been presented in [15].

Much less attention has been paid to numerical approximations of Maxwell equations on general curved domains. However, such domains can easily appear in some applications related to the design of resonant structures, subscreens or waveguides. It is obvious to see that the main difficulties to study these problems arise from the variational crime committed by the approximation of the curved boundaries.

In the recent article [1], a theory for the analysis of numerical approximations of variationally formulated eigenvalue problems posed on general curved domains was presented. It is based on the abstract convergence theory developed in [24] and [25] for conforming discretizations of non compact operators in Banach spaces.
In this paper, applying the results in [1], we analyze the finite element approximations of the Maxwell spectral problem on a curved non convex three-dimensional domain \( \Omega \). More precisely, we prove convergence, optimal order error estimates for approximating eigenfunctions and a double order for eigenvalues for standard Nédélec elements of the first family [37]. These estimates are proved to be valid for any piecewise smooth Lipschitz polyhedral domain for which the eigenfunctions not necessary belong to \( H^1(\Omega) \). As far as we know, these results are new.

The outline of this paper is the following.

In the next section we introduce some preliminary notation and a number of functional spaces, as usual in papers devoted to Maxwell equations.

The continuous problem, the weak formulation widely used to deal with and their discretizations are presented in sections 3 and 4, respectively.

In section 5 we recall some definitions and concepts concerning spectrally correct approximations and summarize the theoretical results obtained in [1].

In section 6 we present some technical results concerning interpolation operators, Helmholtz decompositions and vector potentials defined for vector fields extended by zero outside their original domains.

A condition which is a central point in any application of results in [1] is the convergence of the solution operators in mesh dependent norm. This concept was first introduced in [24]. It is known that the possibility of proving such h-dependent convergence is subject to the validity of a discrete compactness property. As a consequence, the possibility of a good approximation of eigenvalues is subject to the same condition too (see section 4 in [16]).

The discrete compactness property was introduced by Kikuchi [33] and it implies a certain control of the true divergence of the chosen discrete vector fields. It has been extensively studied in the framework of conforming finite element methods where it is well known to hold true for a variety of edge finite elements on quite general two and three dimensional meshes (see, for instance, [33], [7], [8], [35], [14]). Also, the analysis of this property has been carried out in the context of discontinuous Galerkin approximations (see [15]) and within the p- and hp-version for rectangular edge finite elements (see [12]).

In section 7 we investigate the discrete compactness property to include the non conforming case where the approximate domains \( \Omega_h \) are not subdomains of \( \Omega \). We prove it for tetrahedral elements of lowest order.

Finally, in sections 8 and 9 we give optimal order error estimates for the eigenfunctions and the eigenvalues, respectively.
2 Notation

Let $D$ be any bounded domain in $\mathbb{R}^3$ and define the space $L^2(D)^3$ of real, vector valued, square integrable functions $u : D \to \mathbb{R}^3$. We denote by $H^r(D)$ the standard Sobolev space of functions with regular exponents $r$ and norm $\| \cdot \|_{r,D}$, with the convention $H^0 \equiv L^2$. The space $H_0^1(D)$ is the subspace of $H^1(D)$ of functions with zero trace on $\partial D$. We introduce the following spaces:

- $H(\text{curl}; D) := \{ v \in L^2(D)^3 : \text{curl} \, v \in L^2(D)^3 \}$,
- $H_0(\text{curl}; D) := \{ v \in H(\text{curl}; D) : n \times v = 0 \text{ on } \partial D \}$,
- $H_0^0(\text{curl}; D) := \{ v \in H(\text{curl}; D) : \text{curl} \, v = 0 \}$.

These spaces are endowed with the natural graph norm

$$\|v\|_{H(\text{curl}; D)}^2 = \|v\|_{0,D}^2 + \|\text{curl} \, v\|_{0,D}^2.$$ 

We also make use of the spaces

- $H(\text{div}; D) := \{ v \in L^2(D)^3 : \text{div} \, v \in L^2(D)^3 \}$,
- $H_0(\text{div}; D) := \{ v \in H(\text{div}; D) : v \cdot n = 0 \text{ on } \partial D \}$,

and, for each positive real number $r$,

- $H^r(\text{curl}; D) := \{ v \in H^r(D)^3 : \text{curl} \, v \in H^r(D)^3 \}$,
- $H^r(\text{div}; D) := \{ v \in H^r(D)^3 : \text{div} \, v \in H^r(D)^3 \}$,

which is a Hilbert space endowed with the following norm

$$\|v\|_{H^r(\text{curl}; D)}^2 = \|v\|_{r,D}^2 + \|\text{curl} \, v\|_{r,D}^2.$$ 

Next, we introduce spaces and notations depending on a weight function $\rho$. Let $\rho$ be a scalar value function defined on $D$. We assume that $\rho \in L^\infty(D)$ and that there exist positive constants $\rho^*$ and $\rho_*$ such that

$$\rho_* \leq \rho \leq \rho^*,$$

for almost $x$ in $D$. We define the spaces

- $H(\text{div} \rho; D) := \{ v \in L^2(D)^3 : \text{div} \, (\rho v) \in L^2(D) \}$,
- $H(\text{div}^0 \rho; D) := \{ v \in L^2(D)^3 : \text{div} \, (\rho v) = 0 \}$,
- $H_0(\text{div} \rho; D) := \{ v \in H(\text{div} \rho; D) : (\rho v) \cdot n = 0 \text{ on } \partial D \}$,

and the two closed subspaces of $H(\text{curl}; D)$,

$$X_N(D) := H_0(\text{curl}; D) \cap H(\text{div} \rho; D),$$
\[ X_T(D) := H(\text{curl}; D) \cap H_0(\text{div}_\rho; D). \]

According to Proposition 3.7 of [2] and the assumptions on \( \rho \), there exists a regularity exponent \( r \in (1/2, 1] \), depending only on \( D \), such that

\[ X_N(D) \hookrightarrow H^r(D)^3, \]
\[ X_T(D) \hookrightarrow H^r(D)^3. \]

Moreover, for any \( v \in X_N(D) \) or any \( v \in X_T(D) \) it holds

\[ \|v\|_{0,D} \leq C \left( \|\text{curl} \, v\|_{0,D} + \|\text{div} \, \rho \, v\|_{0,D} \right), \]

which follows from Proposition 7.4 of [28].

We denote by \((\cdot, \cdot)\) the standard inner product in \( L^2(D)^3 \) and we write \( L^2_\rho(D)^3 \) for the space \( L^2(D)^3 \) endowed with the following \( \rho \)-weighted inner product

\[ (u, v)_\rho = \int_D \rho u \cdot v. \]

The \( L^2 \)-norm and the \( L^2_\rho \)-norm are clearly equivalent due to the assumptions on \( \rho \).

Given an open set \( D \subset \mathbb{R}^3 \), let \( W(D) \) denote any of the spaces of functions introduced above. We define the restriction operator \( \hat{S} \) by

\[ \hat{S} : W(\mathbb{R}^3) \to W(D) \]
\[ v \mapsto v|_D \]

and the extension operator \( \hat{S} \) by

\[ \hat{S} : W_0(D) \to W(\mathbb{R}^3) \]
\[ v \mapsto \bar{v} \]

where \( \bar{v} \) denotes the extension of a function \( v \) by zero from its original domain \( D \) to \( \mathbb{R}^3 \). From now and on, in order to simplify notation, we will simple write \( \| \cdot \| \) instead of \( \| \cdot \|_{H(\text{curl}; \mathbb{R}^3)} \).

Finally, we define \( X(D) := L^2(D)^3, \ V(D) := H(\text{curl}; D) \) and \( V_0(D) := H_0(\text{curl}; D) \).
3 Continuous problem

Throughout this paper we denote by \( \Omega \) the problem domain which we assume to be a bounded open subset of \( \mathbb{R}^3 \). In general, \( \Omega \) will be non convex and will have a Lipschitz continuous boundary \( \Gamma := \partial \Omega \), which can be equipped with an exterior unit normal vector field \( \mathbf{n} \in L^\infty(\Gamma) \). We also assume that \( \Omega \) is piecewise smooth; more precisely, a curved Lipschitz polyhedral domain (see [19] for definitions) with a connected boundary.

We consider the following Maxwell eigenvalue problem:

\[
\{ \begin{array}{l}
curl(\mu^{-1}\curl u) = \omega^2 \varepsilon u, \quad \text{in } \Omega \\
div(\varepsilon u) = 0, \quad \text{in } \Omega \\
\mathbf{n} \times \mathbf{u} = 0, \quad \text{on } \Gamma.
\end{array} \quad (3.1)
\]

The coefficients \( \varepsilon \) and \( \mu \) are the electric permittivity and the magnetic permeability, respectively. We assume that \( \varepsilon \) and \( \mu \) are real scalar and smooth functions defined on \( \tilde{\Omega} \) and satisfying

\[
0 < \varepsilon_* \leq \varepsilon(x) \leq \varepsilon^*, \quad 0 < \mu_* \leq \mu(x) \leq \mu^*.
\]

(3.2)

The domain \( \tilde{\Omega} \supset \bar{\Omega} \) and will be specified below.

Let us now consider the bilinear and symmetric form \( a \) defined on \( \mathbf{V}_0(\Omega) \times \mathbf{V}_0(\Omega) \) by

\[
a(u, v) := \int_{\Omega} \mu^{-1} \curl u \cdot \curl v + \int_{\Omega} \varepsilon u \cdot v.
\]

Owing to the assumptions about \( \varepsilon \) and \( \mu \), it is easy to check that the form \( a \) is continuous and coercive on \( \mathbf{V}(\Omega) \). Let \( b \) be the bilinear and continuous form defined on \( \mathbf{X}(\Omega) \times \mathbf{X}(\Omega) \) by

\[
b(u, v) := \int_{\Omega} \varepsilon u \cdot v.
\]

Then, the variationally formulation of the spectral problem (3.1) reads as follows:

**Find** \( \omega \in \mathbb{R} \) **and** \( u \neq 0 \) **such that**

\[
a(u, v) = (1 + \omega^2)b(u, v), \quad \forall v \in \mathbf{V}_0(\Omega).
\]

(3.3)
Problem 3.3 has exactly two types of solutions:

- \( \omega = 0 \), with corresponding eigenspace \( K = H_0(\text{curl}^0; \Omega) = \nabla H_0^1(\Omega) \),
- a sequence of finite multiplicity eigenvalues \( \omega_n > 0 \), converging to \(+\infty\), with corresponding eigenfunctions \( u_n \in G = H(\text{div}^0, \Omega) \cap V_0(\Omega) \).

**Remark 3.1** The presence of an eigenspace associated to \( \omega = 0 \) is due to the fact that we have not used the zero divergence constraint to obtain the variational formulation 3.3.

**Remark 3.2** Let us denote by \( K_\perp L_\varepsilon \) and \( K_\perp V_0 \) the orthogonal complements of \( K \) in \( L_\varepsilon^2(\Omega)^3 \) and in \( V_0(\Omega) \) with the inner product \( a(\cdot, \cdot) \), respectively. It is easy to check that

\[
G = K_\perp V_0 = K_\perp L_\varepsilon \cap V_0(\Omega).
\]

In order to analyze the spectral problem (3.3), let us introduce the following linear operator \( T \) defined by

\[
T : X(\Omega) \to V_0(\Omega) \subset X(\Omega)
\]

\[
f \mapsto u,
\]

with \( u \) being the solution of the elliptic problem

\[
a(u, v) = b(f, v), \quad \forall v \in V_0(\Omega).
\] (3.4)

Since \( a \) is elliptic and \( b \) is continuous, by virtue of Lax-Milgram Lemma and the continuity of the inclusion \( V_0(\Omega) \subset X(\Omega) \), we can conclude that \( T \) is a bounded operator. We can also obtain

\[
\|u\|_{H(\text{curl}, \Omega)} \leq C \|f\|_{0, \Omega}.
\] (3.5)

On the other hand, \( T \) is self-adjoint and positive definite with respect to \( a \) and \( b \). It is clear that \( (\omega^2, u) \) is a solution of problem (3.3) if and only if \( \left( \frac{1}{1 - \omega^2}, u \right) \) is an eigenpair of \( T \). However, the operator \( T \) is not compact. In fact, the restriction \( T|_K \) is the identity on the infinite dimensional subspace \( K \subset V_0(\Omega) \).

The following theorem gives an a priori estimate for the eigenvectors of \( T \) not corresponding to \( \omega = 0 \).

**Theorem 3.3** There exist \( r \in (1/2, 1] \) and \( C > 0 \) such that if \( f \in H(\text{div}^0, \Omega) \), then \( u := Tf \in H^r(\text{curl}, \Omega) \) and

\[
\|u\|_{H^r(\text{curl}, \Omega)} \leq C \|f\|_{0, \Omega}.
\] (3.6)
Proof: It is a consequence of the imbedding results of Proposition 3.7 in [2].

4 Finite element discretization

The curved domain $\Omega$ is approximated by a family of domains $\Omega_h$, $h > 0$, with polyhedral boundary $\Gamma_h$. Let $\mathcal{T}_h$ be a standard partition of $\Omega_h$ into tetrahedra such that each vertex of $\Gamma_h$ is also an element of $\partial \Omega$. The index $h$ denotes, as usual, the mesh size of $\mathcal{T}_h$. We assume that the family $\{\mathcal{T}_h\}$ is regular in the sense of the minimal angle condition, i.e., there is a constant $C$ independent of the choice of $\mathcal{T}_h$ such that $\text{vol}(T) \geq C \text{diam}^3(T)$ for all $T \in \mathcal{T}_h$ (see [18], for instance).

We assume that each partition $\mathcal{T}_h$ of $\Omega_h$ has the following properties:

- each vertex of $\Omega_h$ is a vertex of a $T \in \mathcal{T}_h$,
- each $T \in \mathcal{T}_h$ has at least one vertex in the interior of $\Omega_h$,
- any two tetrahedra, $T, T' \in \mathcal{T}_h$ share at most a vertex, a whole side or a whole face.

Let $V_h$, $E_h$ and $F_h$ denote the sets of vertices, edges and faces of the mesh $\mathcal{T}_h$, respectively. We assume that

- $V_h \subset \bar{\Omega}$
- $V_h \cap \Gamma_h \subset \Gamma$
- $E_h$ contains all the points where the boundary $\Gamma$ is not $C^2$
- for all $T \in \mathcal{T}_h$, at most one face of $T$ lies on $\Gamma_h$.

We denote by $\mathcal{T}^\Gamma_h$ the triangular mesh induced by $\mathcal{T}_h$ on the polyhedral surface $\Gamma_h$.

In what follows we will use some notation and definitions introduced in [27]. Consider a $T \in \mathcal{T}_h$ which has a face $S^T_h \in \mathcal{T}^\Gamma_h$ and we call it a boundary tetrahedra. We enumerate the vertices of $T$ such that the vertices of $S^T_h$ are numbered first and we denote them by $P_1^T, P_2^T, P_3^T$ and $P_4^T$, in local notation. Let $\Sigma^T_h$ be the part of $\Gamma$ which is approximated by the face $S^T_h$. We denote by $T^{id}$ the closed tetrahedra with three plane faces, having $P_4$ as a common vertex, and with one curved face, coinciding with $\Sigma^T_h$, and we call it the ideal tetrahedra associated with $T \in \mathcal{T}_h$. For the sake of simplicity, we assume that the partitions $\mathcal{T}_h$ are such that, for each boundary tetrahedra $T$, either $T \subset T^{id}$ or $T \supset T^{id}$. If we replace all boundary tetrahedra in $\mathcal{T}_h$ by their associated ideal tetrahedra $T^{id}$, we obtain the so called ideal partition $\mathcal{T}^{id}_h$ of the domain $\Omega$. 

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For a family \( \{T_h\} \), we denote by \( \hat{\Omega} \) a closed domain in \( \mathbb{R}^3 \) satisfying \( \hat{\Omega} \supset \bar{\Omega} \cup \bar{\Omega}_h \), regardless of \( h \), and we set \( \hat{\Omega}_h := \Omega \cup \Omega_h \).

Finally, to short the notation, we also denote the outer unit normal vector to \( \Gamma_h \) by \( n \).

The vector field \( u \) will be discretized by using the lowest-order Nédélec finite element space:

\[
N_h(\Omega_h) := \{ \mathbf{v}_h \in H(\text{curl}, \Omega_h) : \mathbf{v}_h|_T \in N(\mathbf{T}), \forall T \in T_h \},
\]

where, for each tetrahedron \( T \in T_h \),

\[
N(\mathbf{T}) := \{ \mathbf{v}_h \in P_1(\mathbf{T})^3 : \mathbf{v}_h(x) = \mathbf{a} \times x + \mathbf{b}, \; \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \; x \in T \}.
\]

This space was introduced in [37]. Its elements are piecewise linear vector fields with tangential traces that are continuous through the faces of the mesh. See [29] for a detailed mathematical analysis.

Now, for a given partition \( T_h \) of \( \Omega_h \), we introduce the space

\[
V_h := \{ \mathbf{v}_h \in N_h(\Omega_h) : \mathbf{v}_h \times n|_{\Gamma_h} = 0 \},
\]

which satisfies \( V_h \subset V(\Omega_h) \) for all \( h \).

Let us consider the symmetric bilinear forms

\[
a_h(\mathbf{u}, \mathbf{v}) := \int_{\Omega_h} \mu^{-1} \text{curl} \; \mathbf{u} \cdot \text{curl} \; \mathbf{v} + \int_{\Omega_h} \varepsilon \mathbf{u} \cdot \mathbf{v}, \; \mathbf{u}, \mathbf{v} \in V(\Omega_h),
\]

\[
b_h(\mathbf{u}, \mathbf{v}) := \int_{\Omega_h} \varepsilon \mathbf{u} \cdot \mathbf{v}, \; \mathbf{u}, \mathbf{v} \in X(\Omega_h).
\]

Because of (3.2), it is straightforward to prove that \( a_h \) is continuous and coercive on \( V(\Omega_h) \), uniformly on \( h \).

Then, the discretization of the spectral problem (3.3) is given by

Find \( \omega_h \in \mathbb{R} \) and \( \mathbf{u}_h \in V_h, \mathbf{u}_h \neq 0 \) such that

\[
a_h(\mathbf{u}_h, \mathbf{v}_h) = (1 + \omega_h^2)b_h(\mathbf{u}_h, \mathbf{v}_h), \; \forall \mathbf{v}_h \in V_h.
\]

(4.1)

It can be proved that problem (4.1) has also two kinds of solutions.

• \( \omega_{h0} = 0 \), with corresponding eigenspace \( K_h = H_0(\text{curl}^0; \Omega_h) \cap V_h \).
• a finite set of positive eigenvalues $\omega_{hn} > 0$ with corresponding eigenfunctions $u_{hn} \in G_h$, where $G_h$ is the orthogonal complement of $K_h$ in $V_h$ with respect to $a_h(\cdot, \cdot)$.

**Remark 4.1** Notice that $G_h$ and $K_h$ are also orthogonal in $L^2(\Omega)^3$, i.e.,

$$G_h = \{ v_h \in V_h : (v_h, w_h)_{\varepsilon} = 0, \forall w_h \in K_h \}.$$ 

**Remark 4.2** Since $V_h \not\subseteq V_0(\Omega)$, (4.1) represents a non conforming approximation to (3.3).

5 Spectral approximation

In order to study the convergence properties of problem (4.1), we are going to make use of the theory developed in [1] for non compact operators. In this section, we present the main results obtained in that paper.

Let us consider the spaces $\widetilde{V}(\mathbb{R}^3)$ and $\widetilde{V}_h(\mathbb{R}^3)$ defined by the relations

$$\widetilde{V}(\mathbb{R}^3) = \hat{S}(V) \subset H(\text{curl}; \mathbb{R}^3),$$

$$\widetilde{V}_h(\mathbb{R}^3) = \hat{S}(V_h) \subset H(\text{curl}; \mathbb{R}^3),$$

with $\hat{S}$ being the extension operator defined in section 3. We define the linear operator $A$ by

$$A : X(\mathbb{R}^3) \rightarrow \widetilde{V}(\mathbb{R}^3)$$

$$f \mapsto \bar{u} = \hat{S}T\hat{S}f.$$ 

Clearly, $A$ is also self-adjoint with respect to $a$. Notice that $(\omega^2, u)$ is a solution of problem (3.3) if and only if $(\frac{1}{1+\omega^2}, \bar{u})$ is an eigenpair of $T$ which, in its turn, is equivalent to $(\frac{1}{1+\omega^2}, \bar{u})$ being an eigenpair of $A$, where $\bar{u} = \hat{S}(u)$.

We denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of the solution operator $A$, respectively. For any $z \in \rho(A)$, $R_z(A) = (z - A)^{-1}$ defines the resolvent operator from $\widetilde{V}(\mathbb{R}^3)$ to $\widetilde{V}(\mathbb{R}^3)$.

The discrete analogue of operator $A$ is defined as follows

$$A_h : X(\mathbb{R}^3) \rightarrow \widetilde{V}_h(\mathbb{R}^3)$$

$$f \mapsto \bar{u}_h : \bar{u}_h|_{\Omega_h} = u_h,$$

where $u_h$ is the solution of

$$a_h(u_h, v_h) = b_h(f, v_h), \quad \forall v_h \in V_h. \quad (5.1)$$
Since $a_h$ is continuous and coercive on $V(\Omega_h)$ uniformly on $h$, problem (5.1) is well posed. Then, as a consequence of Lax-Milgram Lemma, the operator $A_h$ is bounded uniformly on $h$ and we have
\[ \|u_h\|_{H^1(\text{curl};\Omega_h)} \leq C\|f\|. \]  
(5.2)

Finally, we denote by $\sigma(A_h)$ and $\rho(A_h)$ the spectrum and the resolvent set of the discrete solution operator $A_h$, respectively.

Let $\lambda$ be a nonzero isolated eigenvalue of $A$ with algebraic multiplicities $m$. Let $\Gamma$ be a circle in the complex plane centered at $\lambda$ which lies in $\rho(A)$ and which encloses no other points of $\sigma(A)$. The continuous spectral projector, $E : V(\mathbb{R}^3) \to \tilde{V}(\mathbb{R}^3)$, relative to $\lambda$ is defined by
\[ E = \frac{1}{2\pi i} \int_{\Gamma} R_z(A) \, dz. \]

We assume the following properties to be satisfied:

**P1:**
\[ \lim_{h \to 0} \|(A - A_h)|_{\tilde{V}_h(\mathbb{R}^3)}\| = 0. \]

**P2:** For each function $x$ of $E(V(\mathbb{R}^3))$,
\[ \lim_{h \to 0} \|x\|_{\Omega;\Omega_h} = 0. \]

**P3:** For each function $x$ of $E(V(\mathbb{R}^3))$,
\[ \lim_{h \to 0} \left( \inf_{x_h \in \tilde{V}_h(\mathbb{R}^3)} \|x - x_h\| \right) = 0. \]

**P4:**
\[ \lim_{h \to 0} \|(A - A_h)|_{E(V(\mathbb{R}^3))}\| = 0. \]

It has been established in [1] that the following theorem is a direct consequence of property P1.

**Theorem 5.1** Let $O \in \mathbb{C}$ be a compact set not intersecting $\sigma(A)$. There exists $h_0 > 0$ such that, if $h < h_0$, then $O$ does not intersect $\sigma(A_h|_{\tilde{V}_h(\mathbb{R}^3)})$.

So, if $h$ is small enough, $\Gamma \subset \rho(A_h|_{\tilde{V}_h(\mathbb{R}^3)})$ and the discrete resolvent operator $R_z(A_h|_{\tilde{V}_h(\mathbb{R}^3)}) = (z - A_h|_{\tilde{V}_h(\mathbb{R}^3)})^{-1}$ is a well defined continuous operator from...
\( \bar{V}_h(\mathbb{R}^3) \) to \( \tilde{V}_h(\mathbb{R}^3) \). Then, we can define the discrete spectral projector \( E_h : V(\mathbb{R}^3) \to \bar{V}_h(\mathbb{R}^3) \) by

\[
E_h = \frac{1}{2\pi i} \int_{\Gamma} R_z(A_h|\bar{V}_h(\mathbb{R}^3)) \, dz.
\]

Let us recall the definition of the gap \( \hat{\delta} \) between two closed subspaces, \( Y \) and \( Z \), of \( V(\mathbb{R}^3) \). We define

\[
\hat{\delta}(Y, Z) := \max\{\delta(Y, Z), \delta(Z, Y)\},
\]

where

\[
\delta(Y, Z) := \sup_{y \in Y} \left( \inf_{z \in Z} \|y - z\| \right).
\]

For eigenspaces associated with \( \omega \neq 0 \), we have the following results.

**Theorem 5.2** Under assumption \( P1 \),

\[
\lim_{h \to 0} \| (E - E_h)|\bar{V}_h(\mathbb{R}^3) \| = 0.
\]

**Theorem 5.3** Under the assumption \( P1 \), for all \( x \in E_h(V(\mathbb{R}^3)) \) there holds

\[
\lim_{h \to 0} \delta(x, E(V(\mathbb{R}^3))) = 0.
\]

**Theorem 5.4** Under the assumptions \( P1 \) and \( P3 \), for all \( x \in E(V(\mathbb{R}^3)) \) holds

\[
\lim_{h \to 0} \delta(x, E_h(V(\mathbb{R}^3))) = 0.
\]

**Theorem 5.5** Under the assumptions \( P1 \) and \( P3 \),

\[
\lim_{h \to 0} \hat{\delta}(E(V(\mathbb{R}^3)), E_h(V(\mathbb{R}^3))) = 0.
\]

As a consequence of the previous theorems, isolated parts of the spectrum of \( A \) are approximated by isolated parts of the spectrum of \( A_h \) (see [32] and [24]). More precisely, for any eigenvalue \( \lambda \) of \( A \) of finite multiplicity \( m \), there exist exactly \( m \) eigenvalues \( \lambda_{1h}, \cdots, \lambda_{mh} \) of \( A_h \), repeated according to their respective multiplicities, converge to \( \lambda \) as \( h \) goes to zero.

Notice that these results show that the numerical method does not introduce spurious modes.

Finally, regarding the eigenvalues, we can establish estimates providing an optimal order of convergence. To do this, we need to introduce other operators.
Let $\Pi_h : V(\mathbb{R}^3) \rightarrow V(\mathbb{R}^3)$ be the projector defined by the relations

$$a_h(x - \Pi_h x, y) = 0, \quad \forall y \in V_h$$

$$|\Pi_h x|_{\mathbb{R}^3 \setminus \Omega_h} = 0.$$  

(5.3)

Because $V_h$ is a closed subset of $V(\Omega_h)$, $(\Pi_h x)|_{\Omega_h} \in V_h$. Hence, we have $\Pi_h x \in \tilde{V}_h(\mathbb{R}^3)$. Notice that the operator $\Pi_h$ is bounded uniformly on $h$ because of the continuity and the coerciveness of $a_h$. On the other hand, since $a_h$ is symmetric, $\Pi_h$ is self-adjoint with respect to $a_h$.

Let us now consider the following consistency terms

$$\gamma_h := \delta(E(V(\mathbb{R}^3)), \tilde{V}_h(\mathbb{R}^3)) + \sup_{y \in E(V(\mathbb{R}^3)) \|y\| = 1} \|y\|_{\Omega \setminus \Omega_h},$$

$$\delta_h := \gamma_h + \|(A - A_h)|_{E(V(\mathbb{R}^3))}\|,$$

$$M_h = \sup_{x \in E(V(\mathbb{R}^3)) \|x\| = 1} \sup_{y \in E(V(\mathbb{R}^3)) \|y\| = 1} |a_h(\Pi_h y - y, \Pi_h y - y)|,$$

$$N_h = \sup_{x \in E(V(\mathbb{R}^3)) \|x\| = 1} \sup_{y \in E(V(\mathbb{R}^3)) \|y\| = 1} |a_h(x, y) - b_h(x, y)|.$$

Notice that from properties P2, P3, and P4, $\delta_h \to 0$ as $h \to 0$.

The following theorem shows how the eigenvalues of $A$ are approximated by those of $A_h$. Its proof is given in [1]

**Theorem 5.6**

i) $|\lambda - \frac{1}{m} \sum_{i=1}^{m} \lambda_{ih}| \leq C(\delta_h^2 + M_h + N_h)$

ii) $\max_{i=1, \ldots, m} |\lambda - \lambda_{ih}| \leq C(\delta_h^2 + M_h + N_h)$

6 Auxiliary results

This section is devoted to the collection of some auxiliary results and concepts which will be required throughout the rest of this article.
6.1 Relations between $\Omega$ and $\Omega_h$

Let $T$ be a boundary tetrahedra. Let $\omega_T$ denote the bounded domain defined by $T^{id}\setminus T$ or $T\setminus T^{id}$, as corresponds. As a consequence of the assumed smoothness of $\Gamma$, i.e., piecewise of class $C^2$, the following estimates hold:

**Lemma 6.1** There exists a constant $C > 0$, not depending on $T$, such that

$$|\omega_T| \leq Ch^3,$$

$$|\Omega \setminus \Omega_h| + |\Omega_h \setminus \Omega| \leq Ch^2.$$

**Proof:** The first estimate is a direct consequence of standard interpolation results (see for instance [27]). The second estimate can be proved easily taking into account that $(\Omega \setminus \Omega_h) \cup (\Omega_h \setminus \Omega) = \bigcup \{\omega_T : T \in T_h\}$. $\square$

Again, let $T$ be a boundary tetrahedra such that $T \subset T^{id}$. Let $S_h^T = T \cap \Gamma_h$ be the boundary face of $T$. We consider the natural extension of the interior faces of $T$, from $S_h^T$ to $\omega_T$, and we denote them by $R_i^T$, $i = 1, 2, 3$. The following surfaces

$$\mathcal{H}_i = R_i^T \cap \omega_T \quad i = 1, 2, 3,$$

which have one face curved, satisfy the following property:

**Lemma 6.2** There exists a constant $C > 0$, independent of $h$, such that

$$|\mathcal{H}_i| \leq Ch^3 \quad i = 1, 2, 3.$$

**Proof:** The proof is identical to that of the Lemma 6.1. $\square$

**Lemma 6.3** There exists a positive constant $C$ such that:

$$\|v\|_{0,\Omega}\leq Ch^s\|v\|_{s,\Omega} \quad \forall v \in H^s(\Omega), \quad 0 \leq s \leq 1,$$

$$\|v\|_{0,\Omega_h}\leq Ch^s\|v\|_{s,\Omega} \quad \forall v \in H^s(\Omega_h), \quad 0 \leq s \leq 1.$$

**Proof:** By adapting the arguments used in the proof of Lemma 3.3.11 in [27] for the three dimensional case, the inequalities can be proved for $s = 1$. Since the two inequalities are clearly true for $s = 0$, they follow for $0 < s < 1$ from standard results on interpolation in Sobolev spaces. $\square$

6.2 Approximation operators

In this section, we introduce standard approximation operators and state their properties. We begin by recalling the definition and some properties of the
curl-conforming Nédélec interpolant.

If \( \mathbf{v} \) is smooth enough, then its Nédélec interpolant \( \Pi_N \mathbf{v} \) is well defined by

\[
\Pi_N \mathbf{v} \in \mathcal{N}_h(\Omega_h) : \int_{\ell} \Pi_N \mathbf{v} \cdot \mathbf{t}_\ell = \int_{\ell} \mathbf{v} \cdot \mathbf{t}_\ell \quad \forall \ell \text{ edge of } \mathcal{E}_h,
\]  

(6.1)

where \( \mathbf{t}_\ell \) denotes a unit vector tangent to the edge \( \ell \). The results of [2] allow extending the above definition to the space \( H^{r_h}(\text{curl}; \Omega_h) \).

Lemma 6.4 The Nédélec interpolation operator

\[
H^2(\Omega_h)^3 \rightarrow \mathcal{N}_h(\Omega_h),
\]

\[\mathbf{v} \mapsto \Pi_N \mathbf{v}\]

extends uniquely to \( H^{r_h}(\text{curl}; \Omega_h) \), with \( r_h > 1/2 \), and the following error estimates holds

\[
\| \mathbf{v} - \Pi_N \mathbf{v} \|_{H(\text{curl}; \Omega_h)} \leq C h^{r_h} \| \mathbf{v} \|_{H^{r_h}(\text{curl}; \Omega_h)}.
\]  

(6.2)

Proof: According to Sobolev Imbedding Theorem and a trace theorem, for each \( T \in \mathcal{T}_h \), we have \( \mathbf{v}|_T \in L^p(T)^3 \), \( \text{curl} \mathbf{v}|_T \in L^p(T)^3 \), and \( \mathbf{v} \times \mathbf{n}|_{\partial T} \in L^p(\partial T)^3 \), with \( p = 4/(3 - 2r_h) > 2 \). Then the result follows by applying Lemma 4.7 of [2]. \( \square \)

We also consider the finite dimensional space \( \mathcal{R}_h(\Omega_h) \) defined by

\[
\mathcal{R}_h(\Omega_h) := \{ \mathbf{v}_h \in H(\text{div}, \Omega_h) : \mathbf{v}_h|_T \in \mathcal{R}(T), \forall T \in \mathcal{T}_h \},
\]

where, for each tetrahedron \( T \in \mathcal{T}_h \),

\[
\mathcal{R}(T) := \{ \mathbf{v}_h \in P_1(T)^3 : \mathbf{v}_h(\mathbf{x}) = \mathbf{a} + \mathbf{b} \mathbf{x}, \mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}, \mathbf{x} \in T \}.
\]

The space \( \mathcal{R}_h(\Omega_h) \) is the lowest-order Raviart-Thomas discretization of \( H(\text{div}, \Omega_h) \). Let us remark that the elements in \( \mathcal{R}_h(\Omega_h) \) have a normal component which is continuous across the interelement boundaries.

Given \( \mathbf{v} \in H(\text{div}, \Omega_h) \cap H^{r_h}(\Omega_h)^3 \), \( r_h > 1/2 \), the Raviart-Thomas interpolant of \( \mathbf{v} \) is defined as the unique \( \mathbf{v}_h \in \mathcal{R}_h(\Omega_h) \) satisfying

\[
\int_f \Pi_R \mathbf{v} \cdot \mathbf{n}_f = \int_f \mathbf{v} \cdot \mathbf{n}_f \quad \forall \text{ face of } \mathcal{F}_h,
\]  

(6.3)

with \( \mathbf{n}_f \) being a unit normal to the face \( f \).
It is well known the following connection between these discrete spaces (see [37]):

$$\text{curl} \mathcal{N}_h(\Omega_h) = \mathcal{R}_h(\Omega_h) \cap H(\text{div}^0; \Omega_h).$$  \hfill (6.4)

The interpolation operators also satisfy the commutativity property

$$\text{curl} \Pi_N \cdot = \Pi_R \text{curl} \cdot$$  \hfill (6.5)

when applied to sufficiently smooth vector fields.

Next, we note that if $v \in H^{r_h}(\text{curl}; \tilde{\Omega}_h)$, then its Nédélec interpolant $\Pi_N v$ is well defined by relation (6.1). However, even if $v|_{\tilde{\Omega}_h} = 0$, $\Pi_N v \notin V_h$ because the tangential trace $n \times (v \times n)$ does not necessarily vanish on the edges of $S^T_h$ corresponding to those boundary tetrahedra $T$ such that $T \subset T^{id}$.

So, we introduce a modified $V_h$-interpolant. Let $\hat{\Pi}_N : H^{r_h}(\text{curl}; \tilde{\Omega}_h) \rightarrow V_h$ be defined in the following way:

$$(\hat{\Pi}_N v)|_{\ell} \cdot t_{\ell} = \begin{cases} 0, & \text{if } \ell \subset \partial S^T_h, \; S^T_h \in T^\Gamma_h : T \subset T^{id} \\ (\Pi_N v)|_{\ell} \cdot t_{\ell}, & \text{otherwise} \end{cases}$$

With this definition we ensure $\hat{\Pi}_N v \in V_h$. In the following lemma we give a bound for the difference between this interpolant and the standard one.

**Lemma 6.5** For each $v \in H^{r_h}(\text{curl}; \tilde{\Omega}_h)$, there exists a positive constant $C$ such that

$$\|\Pi_N v - \hat{\Pi}_N v\|_{H(\text{curl}; \Omega_h)} \leq C h^{r_h} \|v\|_{H^{r_h}(\text{curl}; \Omega_h)}.$$

**Proof:** We have

$$\|\Pi_N v - \hat{\Pi}_N v\|^2_{H(\text{curl}; \Omega_h)} = \sum_{T \in T_h} \|\Pi_N v - \hat{\Pi}_N v\|^2_{H(\text{curl}; T)} = \sum_{S^T_h \in T^\Gamma_h : T \subset T^{id}} \|\Pi_N v - \hat{\Pi}_N v\|^2_{H(\text{curl}; T)}.$$

Let $\varphi_{\ell_i}$ be the standard basis function of $\mathcal{N}_h(\Omega_h)$ associated to an edge $\ell_i \in \mathcal{E}_h$. Let $S^T_h \in T^\Gamma_h$ such that $T \subset T^{id}$. Hence

$$(\Pi_N v - \hat{\Pi}_N v)|_T = \sum_{i=1}^3 \left( \frac{1}{|\ell_i|} \int_{\ell_i} v \cdot t_{\ell_i} \right) \varphi_{\ell_i},$$

where $\bigcup_{i=1}^3 \ell_i = \partial S^T_h$. 

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Now, let \( \omega_T \) denotes the bounded domain defined by \( T^d \setminus T \). Its boundary \( \partial \omega_T \) splits as follows:

\[
\partial \omega_T = \Sigma_h \cup S_h \cup \left( \bigcup_{i=1}^{3} \mathcal{H}_i \right),
\]

with \( \Sigma_h \subset \Gamma \) being the curved side of the tetrahedra \( T^d \) and \( \mathcal{H}_i, i = 1, 2, 3 \), being the curved surfaces such that \( \mathcal{H}_i \cap S_h = \ell_i \).

Then, using Lemma 6.2, we have

\[
\int_{\ell_i} v \cdot t_{\ell_i} = \int_{\mathcal{H}_i} \text{curl } v \cdot n \leq C |\mathcal{H}_i|^{1/2} \| \text{curl } v \|_{0, \mathcal{H}_i} \leq C h^{3/2} \| \text{curl } v \|_{0, \partial \omega_T},
\]

and since \( |\ell_i| \geq Ch \),

\[
\sum_{i=1}^{3} \frac{1}{|\ell_i|} \int_{\ell_i} v \cdot t_{\ell_i} \leq C h^{1/2} \| \text{curl } v \|_{0, \partial \omega_T}.
\]

On the other hand, by using a suitable trace theorem, we have

\[
\| \text{curl } v \|_{0, \partial \omega_T} \leq C \left( h^{-1/2} \| \text{curl } v \|_{0, \omega_T} + h^{r_h-1/2} \| \text{curl } v \|_{r_h, \omega_T} \right).
\]

Hence, Lemma 6.3 yields

\[
\| \Pi_N v - \tilde{\Pi}_N v \|_{H(\text{curl}; \Omega_h)}^2 \leq C \sum_{S_h \subset T^d, T \subset T^d} \left( \| \text{curl } v \|_{0, \omega_T}^2 + h^{2r_h} \| \text{curl } v \|_{r_h, \omega_T}^2 \right) \leq C \left( \| \text{curl } v \|_{0, \Omega_h}^2 + h^{2r_h} \| \text{curl } v \|_{r_h, \Omega_h}^2 \right) \leq C h^{2r_h} \| \text{curl } v \|_{r_h, \Omega_h}^2,
\]

where we have used that \( \| \varphi \|_{H(\text{curl}; \Omega_h)} \leq C \), for a constant \( C \) only depending on the regularity of \( T \). Thus, we conclude the proof. \( \Box \)

Now, for the same partition \( T_h \) of \( \Omega_h \), we introduce the space

\[
W_h := \{ w_h \in \mathcal{R}_h(\Omega_h) : w_h \cdot n|_{\Gamma_h} = 0 \}.
\]

In addition to operator \( \tilde{\Pi}_N \), we also define a \( W_h \)-interpolant. Let \( \tilde{\Pi}_R : H^r(\bar{\Omega}_h) \cap H_0(\text{div}; \bar{\Omega}_h) \to W_h \) be defined in the following way:

\[
(\tilde{\Pi}_R w)|_{\ell} \cdot n_\ell = \begin{cases} 0, & \text{if } \ell \subset \partial S_h^T, S_h^T \in T_h^T : T \subset T^d \\ (\Pi_R w)|_{\ell} \cdot n_\ell, & \text{otherwise} \end{cases}
\]
With this definition we ensure $\tilde{\Pi}_R w \in W_h$ and we get
\[
\text{curl} \tilde{\Pi}_N \cdot = \tilde{\Pi}_R \text{curl} \cdot \tag{6.6}
\]

Finally, let $\tilde{\Pi}_L$ be the standard linear interpolation operator defined on continuous functions $v$ with domain $\hat{\Omega}_h$. Notice that if $v|_{\partial \hat{\Omega}_h} = 0$, properties of $T_h$ lead to $\tilde{\Pi}_L v|_{\Gamma_h} = 0$ and we have
\[
\nabla \tilde{\Pi}_L \cdot = \tilde{\Pi}_N \nabla \cdot \tag{6.7}
\]

Relations (6.6) and (6.7) follow from the definitions of the interpolation operators and the theorems of Green and Stokes.

### 6.3 Helmholtz decompositions

We begin by recalling that the continuous Helmholtz decomposition
\[
L^2(\Omega)^3 = H(\text{div}_e^0; \Omega) \oplus H_0(\text{curl}^0; \Omega) \tag{6.8}
\]
holds and the subspaces of the right hand side are closed and $\varepsilon$-orthogonal in $L^2(\Omega)^3$ (see [28]).

A similar decomposition holds on the discrete level. We have
\[
V_h = \mathcal{Y}_h \oplus \nabla \mathcal{Z}_h, \tag{6.9}
\]
where the spaces $\mathcal{Y}_h$ and $\mathcal{Z}_h$ are given by
\[
\mathcal{Z}_h := \{ p \in H^1_0(\Omega_h) : p|_T \in \mathcal{P}_1(T), T \in T_h \},
\]
with $\mathcal{P}_1(T)$ denoting the space of polynomials of degree at most one restricted to $T$, and
\[
\mathcal{Y}_h := \{ v \in V_h : (v, \nabla p) = 0 \ \forall p \in \mathcal{Z}_h \}.
\]

The space $\mathcal{Y}_h$ is referred to as the space of discretely divergence free functions. By construction, the above discrete decomposition is orthogonal in $L^2(\Omega_h)^3$ and in $H(\text{curl}; \Omega_h)$ (see [3]).

The family of subspaces $\{ \mathcal{Z}_h \}$ satisfy the following property:

**Property 6.6** For each $p \in H^1_0(\Omega)$, holds
\[
\lim_{h \to 0} \inf_{p_h \in \mathcal{Z}_h} \| \bar{p} - p_h \|_{1, \mathbb{R}^3} = 0.
\]
Proof: The proof follows directly by using density arguments, standard interpolation results and the inequalities in Lemma 6.3 to handle the terms appearing because of the discrepancy between $\Omega$ and $\Omega_h$. □

Remark 6.7 Since $H_0(\text{curl}; \mathbb{R}^3)$ differs from $L^2(\mathbb{R}^3)$ only by the addition of rotor terms, 

$$\inf_{p_h \in Z_h} \| \nabla \bar{p} - \nabla \tilde{p}_h \| = \inf_{p_h \in Z_h} \| \nabla \bar{p} - \nabla \tilde{p}_h \|_{0, \mathbb{R}^3}.$$ 

According to Lemma 5.6 in [29], the subspace $\nabla Z_h$ coincides with the discrete kernel $K_h$ defined in section 4. Then, Property 6.6 can be rephrased as follows: for each $v \in K$, holds 

$$\lim_{h \to 0} \inf_{v_h \in K_h} \| \nabla \bar{v} - \nabla \tilde{v}_h \| = 0.$$ 

This property is called the completeness of the discrete kernel condition (see [16] for further discussion) and it is well known to hold true when $\Omega_h = \Omega$.

Lemma 6.8 For any $w_h \in G_h$, $\bar{w}_h = \hat{S}(w_h)$ has the discrete Helmholtz decomposition in $\hat{\Omega}_h$

$$\bar{w}_h = \nabla \xi + \epsilon^{-1} \text{curl} \varphi,$$

with $\varphi \in H(\text{div}^0; \hat{\Omega}_h)$. Moreover, there exist positive constants $C$ and $r_h \in (1/2, 1]$, such that

$$\| \text{curl} \varphi \|_{r_h, \hat{\Omega}_h} \leq C \| w_h \|_{H(\text{curl}; \Omega_h)}$$ (6.10)

$$\| \epsilon^{1/2} \nabla \xi \|_{0, \hat{\Omega}_h} \leq C h^{r_h} \| w_h \|_{H(\text{curl}; \Omega_h)}.$$ (6.11)

Proof: Let $\xi \in H_0^1(\hat{\Omega}_h)$ be the unique solution of the Dirichlet problem

$$\begin{cases}
\text{div} (\epsilon \nabla \xi) = \text{div} (\epsilon \bar{w}_h), & \text{in } \hat{\Omega}_h \\
\xi = 0, & \text{on } \partial \hat{\Omega}_h.
\end{cases}$$

Hence, $\epsilon(\bar{w}_h - \nabla \xi)$ is a divergence-free vector of $H(\text{div}; \hat{\Omega}_h)$ satisfying $\epsilon(\bar{w}_h - \nabla \xi) \cdot n = 0$ in $H^{-1/2}(\partial \hat{\Omega}_h)$. So, there exists a function $\varphi \in H(\text{div}; \hat{\Omega}_h) \cap H(\text{curl}; \hat{\Omega}_h)$ such that

$$\text{curl} \varphi = \epsilon(\bar{w}_h - \nabla \xi) \quad \text{in } \hat{\Omega}_h,$$

with the following properties

$$\text{div} \varphi = 0, \quad \text{in } \hat{\Omega}_h$$

$$\varphi \cdot n = 0, \quad \text{on } \partial \hat{\Omega}_h.$$
Now, since the components of $\nabla \xi \times n|_{\partial \Omega_h}$ are the tangential derivatives of the trace $\xi|_{\partial \Omega_h}$, we have $\nabla \xi \times n = 0$ on $\partial \Omega_h$. Then,

$$\text{curl} \varphi \times n = \varepsilon (\hat{w}_h - \nabla \xi) \times n = 0, \quad \text{on } \partial \Omega_h.$$  

So, as a consequence of Proposition 3.7 of [2], we have that $\varphi$ and $\text{curl} \varphi \in H^{r_h}(\tilde{\Omega}_h)$, for some $r_h > 1/2$, and

$$\|\text{curl} \varphi\|_{H^{r_h}(\text{curl}; \tilde{\Omega}_h)} \leq C \|w_h\|_{H(\text{curl}; \Omega_h)}.$$  

On the other hand, we have

$$\int_{\Omega_h} \varepsilon |\nabla \xi|^2 = \int_{\Omega_h} \varepsilon \nabla \xi \cdot (w_h - \varepsilon^{-1} \text{curl} \varphi) - \int_{\Omega_h} \nabla \xi \cdot \text{curl} \varphi$$

$$= \int_{\Omega_h} \varepsilon \nabla \xi \cdot (w_h - \Pi_N(\varepsilon^{-1} \text{curl} \varphi))$$

$$+ \int_{\Omega_h} \varepsilon \nabla \xi \cdot (\Pi_N(\varepsilon^{-1} \text{curl} \varphi) - \varepsilon^{-1} \text{curl} \varphi) - \int_{\Omega \setminus \Omega_h} \nabla \xi \cdot \text{curl} \varphi.$$

Now, put $z_h = w_h - \Pi_N(\varepsilon^{-1} \text{curl} \varphi)$. Using the commutativity property (6.6), we get

$$\text{curl} z_h = \text{curl} w_h - \Pi_R(\varepsilon^{-1} \text{curl} \varphi) = \text{curl} w_h - \Pi_R \text{curl} w_h$$

and, since $\text{curl} w_h \in W_h$, we obtain

$$\text{curl} z_h = 0, \quad \text{in } \Omega_h.$$  

Furthermore,

$$z_h \times n = (w_h - \Pi_N(\varepsilon^{-1} \text{curl} \varphi)) \times n = 0, \quad \text{on } \Gamma_h.$$  

Hence $z_h \in H_0(\varepsilon; \Omega_h)$. Then, since $w_h \in G_h$ and $\text{curl} \varphi \in H(\text{div}; \tilde{\Omega}_h)$, we have

$$\int_{\Omega_h} \varepsilon \nabla \xi \cdot z_h = \int_{\Omega_h} (\varepsilon w_h - \text{curl} \varphi) \cdot z_h = \int_{\Omega_h} \varepsilon w_h \cdot z_h - \int_{\Omega_h} \text{curl} \varphi \cdot \tilde{z}_h = 0.$$  

For the second term, we get

$$\int_{\Omega_h} \varepsilon \nabla \xi \cdot (\Pi_N(\varepsilon^{-1} \text{curl} \varphi) - \varepsilon^{-1} \text{curl} \varphi) \leq C h^{r_h} \varepsilon^{-1} \|\nabla \xi\|_{0, \Omega_h} \|\text{curl} \varphi\|_{r_h, \Omega_h}$$

$$\leq C h^{r_h} \|\nabla \xi\|_{0, \tilde{\Omega}_h} \|\text{curl} \varphi\|_{r_h, \tilde{\Omega}_h}.$$  

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For the last term, we have
\[
\left| \int_{\Omega} \nabla \xi \cdot \text{curl} \varphi \right| \leq C \|\nabla \xi\|_{0,\Omega_0} \|\text{curl} \varphi\|_{0,\Omega_0}
\]
\[
\leq C h^r \|\nabla \xi\|_{0,\hat{\Omega}_h} \|\text{curl} \varphi\|_{r,\hat{\Omega}_h}.
\]
Thus, we conclude the proof. □

**Definition 6.9** Let \( w_h \in Z_h \). A function \( \hat{w} \in H^1_0(\Omega) \) is called to be associated with \( w_h \) if it has the following properties:

- \( \hat{w} \in C^0(\bar{\Omega}) \)
- \( \hat{w}(P_i) = w_h(P_i), \ \forall P_i \in V_h \)
- \( \hat{w} \) is linear on each tetrahedra \( T \in T_h \cap T_id \)
- if \( T \subset T^id \), \( \hat{w} = 0 \) on \( T^id \setminus T \) and \( \hat{w} = w_h \) on \( T \)
- if \( T^id \subset T \), \( \hat{w}|_{\partial T^id \subset \partial \Omega} = 0 \).

The definition above is due to Feistauer and Ženšík (see [27]). The construction of such a function follows basically from the interpolation theory developed to Zlámal [42] for two dimensional curved finite elements. The extension of his ideas to the three dimensional case is relatively straightforward so we do not include the details here.

**Lemma 6.10** Let \( \hat{w} \in H^1_0(\Omega) \) be associated with \( w_h \in Z_h \). Let \( T^id \in T_h^id \) lie along \( \Gamma \) and let \( T \in T_h \) be its approximation. If \( T^id \subset T \), then
\[
\|\hat{w} - w_h\|_{1,T^id} \leq C h\|w_h\|_{1,T},
\]
where \( C \) is a constant independent of \( h \).

**Proof:** The proof is a consequence of Definition 6.9 and a suitable extension of Theorem 2 in [42]. □

### 6.4 Some extension results

In what follows we will use smooth extensions of functions originally defined in \( \Omega \). We denote by \( \varphi^e \) an extension of \( \varphi \) from \( H^s(\Omega) \), \( s > 0 \), into \( H^s(\mathbb{R}^3) \) satisfying \( \varphi^e \in H^s(\mathbb{R}^3) \) and
\[
\|\varphi^e\|_{s,\mathbb{R}^3} \leq C\|\varphi\|_{s,\Omega}
\]  \hspace{1cm} (6.12)

(see [30], for instance).
In [6], it is shown that a function \( v \in X_T(\Omega) \cap H(\text{div}^0_\rho; \Omega) \) can be split in the following way

\[
v = w + \nabla q,
\]

(6.13)

where \( w \in H^1(\Omega)^3 \cap X_T(\Omega) \) and \( q \in H^1(\Omega) \) satisfies

\[
\Delta q \in L^2(\Omega), \quad \frac{\partial q}{\partial n} = 0 \text{ on } \partial \Omega,
\]

with estimates

\[
\|w\|_{1,\Omega} + \|\Delta q\|_{0,\Omega} \leq C \|v\|_{H(\text{curl},\Omega)}.
\]

(6.14)

In the splitting (6.13) of \( v \), the potential \( q \) actually belongs to \( H^{3/2+\delta}(\Omega) \), for some \( \delta > 0 \) (see [20]).

Let \( w^e \) and \( q^e \) denote extensions to \( \mathbb{R}^3 \) of the functions \( w \) and \( q \), respectively, satisfying (6.12). We define

\[
v^e = w^e + \nabla q^e \in H^r(\mathbb{R}^3)^3,
\]

(6.15)

where \( r = 1/2 + \delta \). We have

\[
\|v^e\|_{r,\mathbb{R}^3} = \|w^e + \nabla q^e\|_{r,\mathbb{R}^3} \leq \|w^e\|_{r,\mathbb{R}^3} + \|\nabla q^e\|_{r,\mathbb{R}^3}
\]

(6.16)

\[
\leq C\left(\|w\|_{r,\Omega} + \|\nabla q\|_{r,\Omega}\right) \leq C\left(\|w\|_{1,\Omega} + \|\Delta q\|_{0,\Omega}\right)
\]

\[
\leq C\|v\|_{H(\text{curl},\Omega)}
\]

and

\[
\|\text{curl} v^e\|_{0,\mathbb{R}^3} = \|\text{curl} w^e\|_{0,\mathbb{R}^3} \leq \|w^e\|_{1,\mathbb{R}^3} \leq C \|w\|_{1,\Omega}
\]

(6.17)

\[
\leq C\|v\|_{H(\text{curl},\Omega)}.
\]

Now, let \( u \) be an analytical solution of problem (3.4). Hence, \( u \) is the solution (in the sense of distribution) of the following source problem:

\[
\begin{cases}
\text{curl} (\mu^{-1} \text{curl} u) + \varepsilon u = \varepsilon f, & \text{in } \Omega \\
\mathbf{n} \times u = 0, & \text{on } \Gamma.
\end{cases}
\]

(6.18)

Let \( f \in H(\text{div}^0_\rho; \Omega) \). Then, \( \text{div} \varepsilon u = 0 \) and \( u \) is an element of the space \( X_N(\Omega) \). On the other hand, because of \( \mathbf{u} \times \mathbf{n} = 0 \) on \( \Gamma \), \( \text{curl} u \cdot \mathbf{n} = 0 \) on
\( \Gamma \) too. Hence, \( \mu^{-1}\text{curl } u \) belongs to \( X_T(\Omega) \). Finally, by assumption (3.2), we obtain \( \text{curl } \text{curl } u \in L^2(\Omega)^3 \) and \( \text{curl } u \in X_T(\Omega) \).

So, let \( u^e \) denote an extension to \( \mathbb{R}^3 \) of \( u \) satisfying (6.12) and let \( (\text{curl } u)^e \) denote an extension to \( \mathbb{R}^3 \) of \( \text{curl } u \) as defined in equation (6.15). Then, from estimates (3.6), (6.16), (6.17) and using (6.18), we find that

\[
\|u^e\|_{r, \mathbb{R}^3} \leq C \|u\|_{r, \Omega} \leq C \|f\|_{0, \Omega},
\]

\[
\|(\text{curl } u)^e\|_{r, \mathbb{R}^3} \leq C\|\text{curl } u\|_{H(\text{curl}; \Omega)} \leq C \left( \|f\|_{0, \Omega} + \|u\|_{H(\text{curl}; \Omega)} \right) \leq C \|f\|_{0, \Omega},
\]

\[
\|\text{curl } (\text{curl } u)^e\|_{0, \mathbb{R}^3} \leq C \|\text{curl } u\|_{H(\text{curl}; \Omega)} \leq C \|f\|_{0, \Omega}.
\]

### 6.5 Vector potential in \( \hat{\Omega}_h \)

Let \( \varepsilon \hat{u} \) be a vector field in \( L^2(\hat{\Omega}_h)^3 \) satisfying

\[
\text{div } \varepsilon \hat{u} = 0 \quad \text{in} \quad \hat{\Omega}_h.
\]

Then, among the vector potentials \( \hat{\phi} \) verifying

\[
\text{curl } \hat{\phi} = \varepsilon \hat{u} \quad \text{in} \quad \hat{\Omega}_h
\]

\[
\text{div } \hat{\phi} = 0 \quad \text{in} \quad \hat{\Omega}_h,
\]

we can choose \( \hat{\phi} \in H(\text{curl}; \hat{\Omega}_h) \) such that

\[
\hat{\phi} \cdot n = 0 \quad \text{on} \quad \partial \hat{\Omega}_h.
\]

We can make \( \hat{u} \) be closely related to \( u \) by characterizing \( \hat{\phi} \) as the solution of the following boundary value problem:

\[
\begin{align*}
-\Delta \hat{\phi} &= \varepsilon (\text{curl } u)^e + \nabla \varepsilon \times u^e + \hat{c}, & \text{in } \hat{\Omega}_h \\
\text{curl } \hat{\phi} \times n &= 0, & \text{on } \partial \hat{\Omega}_h
\end{align*}
\]

where the constant vector \( \hat{c} \) is taken for the problem to be compatible (see Remark 6.12 below); i.e.,

\[
\hat{c} := -\frac{1}{|\hat{\Omega}_h|} \int_{\hat{\Omega}_h} \varepsilon (\text{curl } u)^e + \nabla \varepsilon \times u^e = -\frac{1}{|\hat{\Omega}_h|} \int_{\Omega \setminus \hat{\Omega}_h} \varepsilon (\text{curl } u)^e + \nabla \varepsilon \times u^e.
\]
Using Lemmas 6.1 and 6.3 and estimates (6.19) and (6.20), we have

\[
|\hat{c}| \leq \frac{\Omega_h \setminus \Omega^{1/2}}{\Omega_h} \|\varepsilon (\text{curl } u) + \nabla \varepsilon \times u \|_{0, \Omega_h \setminus \Omega} \leq C h^{1+r} \|f\|_{0, \Omega}.
\]  

(6.24)

Now, since \( \hat{\phi} \in X_T(\hat{\Omega}_h) \), Corollary 3.16 of [2] yields

\[
\|\hat{\phi}\|_{H(\text{curl}, \hat{\Omega}_h)} \leq C \|\text{curl } \hat{\phi}\|_{0, \hat{\Omega}_h}.
\]

Hence, from (6.22), we obtain

\[
\|\hat{\phi}\|_{H(\text{curl}, \hat{\Omega}_h)} \leq C \|\varepsilon u\|_{H(\text{curl}, \hat{\Omega}_h)} + \|u\|_{r, \Omega} + \|\hat{c}\|_{\hat{\Omega}_h}^{1/2} \leq C \|f\|_{0, \Omega},
\]

the latter because of (6.19), (6.20) and (6.24).

On the other hand, since \( \hat{\Omega}_h \) is a Lipschitz polyhedron, we have \( \hat{\phi} \in H^{r_h}(\hat{\Omega}_h) \), \( r_h > 1/2 \), and

\[
\|\hat{\phi}\|_{r_h, \hat{\Omega}_h} \leq C \|f\|_{0, \Omega}
\]

(6.26)
as a consequence of Proposition 3.7 of [2].

Finally, since \( \text{curl } \hat{\phi} \in X_N(\hat{\Omega}_h) \), by applying the same Proposition, we have \( \text{curl } \hat{\phi} \in H^{r_h}(\hat{\Omega}_h) \), \( r_h > 1/2 \), and

\[
\|\text{curl } \hat{\phi}\|_{r_h, \hat{\Omega}_h} \leq C \|f\|_{0, \Omega}.
\]

(6.27)

As a consequence of the above estimates, we obtain

\[
\|\varepsilon \hat{u}\|_{H^{r_h}(\text{curl}, \hat{\Omega}_h)} \leq C \|f\|_{0, \Omega}.
\]

(6.28)

The following lemma shows that \( \hat{u} \) is an accurate approximation of \( u \) in \( \Omega \).

**Lemma 6.11**  There exists a positive constant \( C \) such that

\[
\|\varepsilon (u - \hat{u})\|_{H(\text{curl}, \Omega)} \leq C h^{r_h} \|f\|_{0, \Omega}.
\]

**Proof:** Let \( \phi \) be the vector potential associated with \( u \). It is known that \( \phi \)
can be taken to satisfy the following equations

\[
\begin{align*}
-\Delta \phi &= \text{curl} \varepsilon u, \quad \text{in } \Omega \\
\text{div} \phi &= 0, \quad \text{in } \Omega \\
\phi \cdot n &= 0, \quad \text{on } \Gamma \\
\text{curl} \phi \times n &= 0, \quad \text{on } \Gamma
\end{align*}
\] (6.29)

(see Remark 6.12), and we have

\[\|\phi\|_{H^r(\text{curl}; \Omega)} \leq C \|f\|_{0,\Omega}.\] (6.30)

Now, from the definitions of \(\phi\) and \(\hat{\phi}\) we can deduce

\[\varepsilon(\bar{u} - \hat{u}) = \text{curl} \hat{\phi} - \text{curl} \phi = \text{curl}(\phi - \hat{\phi}).\]

Hence, \(\varepsilon(\bar{u} - \hat{u}) \in X_N(\hat{\Omega}_h)\) and we can apply Proposition 7.4 of [28] to obtain

\[\|\varepsilon(\bar{u} - \hat{u})\|_{0,\hat{\Omega}_h} \leq C \|\text{curl} \varepsilon(\bar{u} - \hat{u})\|_{0,\hat{\Omega}_h}.\]

On the other hand, by using estimate (6.28) we have

\[
\|\text{curl} \varepsilon(\bar{u} - \hat{u})\|_{0,\hat{\Omega}_h}^2 = \|\text{curl} \varepsilon(u - \hat{u})\|_{0,\Omega}^2 + \|\text{curl} \varepsilon \hat{u}\|_{0,\Omega_h \setminus \Omega}^2 \\
\leq C \left(\|\text{curl} \varepsilon(u - \hat{u})\|_{0,\Omega}^2 + h^{2r_N} \|\text{curl} \varepsilon \hat{u}\|_{r_N,\hat{\Omega}_h}^2\right) \\
\leq C \left(\|\text{curl} \varepsilon(u - \hat{u})\|_{0,\Omega}^2 + h^{2r_N} \|f\|_{0,\hat{\Omega}_h}^2\right).
\]

From equations (6.22) and (6.29), we have

\[\|\text{curl} \varepsilon(u - \hat{u})\|_{0,\Omega} = |\hat{c}|\Omega|^{1/2} \leq C h^{1+r_N} \|f\|_{0,\Omega}.\]

So, combining the above inequalities, we have the desired result. \(\Box\)

**Remark 6.12** This problem is well-posed. We present the main arguments of the proof by following the ideas given in [20] closely. We introduce the following non-symmetric weak formulation: find \(u \in X_T \cap H(\text{div}^0; \Omega)\) such that

\[
\int_{\hat{\Omega}} \text{curl} u \cdot \text{curl} v = \int_{\hat{\Omega}} f \cdot v, \quad \forall v \in H_T = X_T \cap H^1(\Omega)^3.
\] (6.31)

By using \(C^\infty\) functions in \(H_T\) as test functions and proceeding in the standard way, we see that the problem (6.31) has the following strong form:
\[-\triangle \mathbf{u} = \mathbf{f}, \text{ in } \Omega \tag{6.32}\]
\[\text{div} \mathbf{u} = 0, \text{ in } \Omega \tag{6.33}\]
\[\mathbf{u} \cdot \mathbf{n} = 0, \text{ on } \Gamma \tag{6.34}\]
\[\text{curl} \mathbf{u} \times \mathbf{n} = 0, \text{ on } \Gamma. \tag{6.35}\]

Here, the vector Laplace equation is understood in the sense of distribution in \(\mathbb{R}^3\). The free divergence constrain (6.33) and the boundary condition (6.34) correspond to the definition of the variational spaces. The natural boundary condition (6.35) is obtained after integration by parts and has to be understood in a weak sense.

Taking into account that \(C^\infty\) functions in \(\mathbf{H}_T\) that vanish in a neighborhood of the singular parts of the boundary are dense in \(\mathbf{H}_T\) (see [21]), it is easy to see that for \(\mathbf{f} \in \mathbf{L}^2(\Omega)^3\) and \(\mathbf{u} \in \mathbf{X}_T \cap \mathbf{H}(\text{div}^0; \Omega)\), the weak formulation and the boundary formulation are completely equivalent. Note that, due to (6.35), the source \(\mathbf{f}\) has to satisfy a Neumann-type compatibility condition:

\[
\int_\Omega \mathbf{f} = -\int_\Omega \triangle \mathbf{u} = \int_\Omega \text{curl} \text{ curl} \mathbf{u} = -\int_{\partial\Omega} \text{curl} \mathbf{u} \times \mathbf{n} = 0.
\]

By taking \(\mathbf{v} = \mathbf{u}\) in (6.31) and considering decomposition (6.13) with estimate (6.14), it is easy to see that:

\[
\int_\Omega \text{curl} \mathbf{u} \cdot \text{curl} \mathbf{v} = \| \text{curl} \mathbf{u} \|^2_{0, \Omega} \geq C \| \mathbf{u} \|^2_{H(\text{curl}; \Omega)}
\geq C \| \mathbf{u} \|_{\mathbf{X}_T}(\| \mathbf{v} \|_{1, \Omega} + \| \triangle q \|_{0, \Omega})
\geq C \| \mathbf{u} \|_{\mathbf{X}_T} \| \mathbf{v} \|_{\mathbf{H}_T},
\]

where \(C\) is a positive constant. It is obvious that the bilinear form is continuous. Therefore, we obtain the existence and uniqueness of solution of the variationally formulation (6.31).

**Remark 6.13** As a consequence of Lemma 6.11 and the assumptions on \(\varepsilon\), the following estimate can also be proved:

\[
\| \mathbf{u} - \hat{\mathbf{u}} \|_{H(\text{curl}; \Omega)} \leq C H^{r_n} \| \mathbf{f} \|_{0, \Omega}. \tag{6.36}
\]

In fact, a simple manipulation based on vector identities gives

\[
\| \mathbf{u} - \hat{\mathbf{u}} \|_{H(\text{curl}; \Omega)} \leq \frac{1}{\varepsilon} \| \varepsilon (\mathbf{u} - \hat{\mathbf{u}}) \|_{0, \Omega} + \frac{1}{\varepsilon} \| \text{curl} \varepsilon (\mathbf{u} - \hat{\mathbf{u}}) \|_{0, \Omega}
+ \frac{1}{\varepsilon^2} \| \nabla \varepsilon \times \varepsilon (\mathbf{u} - \hat{\mathbf{u}}) \|_{0, \Omega} \leq \max \left( \left\| \frac{1}{\varepsilon} \right\|, \left\| \nabla \left( \frac{1}{\varepsilon} \right) \right\| \right) \| \varepsilon (\mathbf{u} - \hat{\mathbf{u}}) \|_{H(\text{curl}; \Omega)}
\leq C \| \varepsilon (\mathbf{u} - \hat{\mathbf{u}}) \|_{H(\text{curl}; \Omega)}.
\]

26
7 Discrete compactness property

The discrete compactness property is known to be a sufficient and in a sense also a necessary condition for the convergence of Maxwell eigenvalues without the appearance of any spurious modes (see [16]).

Within the $h$-version, it is proved that the Nédélec family satisfies this property (see [33], [7], [35], [8], [16], [14]). Within the $p$- and $hp$-version, the discrete compactness property is proved for rectangular edge finite elements in the recent paper [12].

In this section, we extend the proof that this property holds to include standard edge elements of the lowest order based on a tetrahedral mesh of $\Omega_h$ which represents an external approximation of $\Omega$; i.e., $\Omega_h \not\subset \Omega$ (see [39] and the references therein for the terminology).

To begin with, we give the definition of discrete compactness property:

**Property 7.1** For any sequence $\{v_h\}$ of discrete divergence free fields $v_h \in G_h$, bounded in $V_h$, there exists a subsequence, still denoted by $\{v_h\}$, and a function $w \in L^2(\Omega)^3$ such that

$$\lim_{h \to 0} \|v_h - w\|_{0,\mathbb{R}^3} = 0.$$

It is known that the discrete compactness property is a consequence of the following condition:

**Property 7.2** There exists a sequence $\eta_h$, $\eta_h \to 0$ when $h \to 0$, such that

$$\forall v_h \in G_h, \exists v \in G : \|v - v_h\| \leq \eta_h \|v_h\|.$$

Moreover, we have the following result, analogous to Proposition 7.13 in [15]:

**Lemma 7.3** Property 7.2 is equivalent to Property 7.1 with strong limit in $G$.

**Proof:**

Property 7.2 $\Rightarrow$ Property 7.1: the proof is identical to that given in the conforming case (i.e., when $\Omega_h = \Omega$) and basically exists in several papers; see, for instance, [8], [16], [13].

Property 7.1 $\Rightarrow$ Property 7.2: the proof is identical to that of Proposition 7.13 in [15].
According to the results in [16], the discrete compactness property for constant material coefficients implies the corresponding discrete compactness property for the case of general material coefficients. An analogous result can be proved for the non conforming finite element methods we are considering:

**Property 7.4** For a given sequence \( \{v_h\} \), the discrete compactness property holds for any \( \mu \) and \( \varepsilon \) if and only if it holds for \( \mu = 1 \) and \( \varepsilon = 1 \).

**Proof:** The proof follows exactly the same arguments given in the proof of Proposition 2.27 in [16] for the conforming case, when \( \Omega = \Omega_h \), in combination with the following result: for each \( u \in G \) and each \( p_h \in Z_h \)

\[
\int_{\Omega \cup \Omega_h} \varepsilon \cdot \nabla \bar{p}_h = \int_{\Omega} \varepsilon \cdot \nabla \bar{p} + \int_{\Omega \cup \Omega_h} \varepsilon \cdot \nabla (\bar{p}_h - \bar{p}) = \sum_{T^{id} \in T_h \atop \partial T^{id} \cap \Gamma \neq \emptyset} \int_{T^{id}} \varepsilon \cdot \nabla (\bar{p} - p_h),
\]

where \( \bar{p} \) denotes the function associated with \( p_h \) as in Definition 6.9. \( \square \)

By virtue of these results, it only remains to prove Property 7.2 for \( \varepsilon = \mu = 1 \). To this end, let us introduce the subspace of divergence free fields:

\[
Y(\Omega) := \{ v \in V_0(\Omega) : \text{div} \, v = 0 \},
\]

which is compactly embedded in \( L^2(\Omega)^3 \). The discrete analogue is the space \( Y_h \), appearing in the discrete Helmholtz decomposition (6.9). Now, we rephrase Property 7.2 as follows:

**Property 7.5** There exists a sequence \( \delta_h, \delta_h \to 0 \) when \( h \to 0 \), such that

\[
\forall v_h \in \mathcal{Y}_h, \, \exists v \in Y(\Omega) : \| \bar{v} - \bar{v}_h \| \leq \delta_h \| \bar{v}_h \|.
\]

**Proof:** The proof of this property will be carried out in three steps and it essentially follows the approach given in [14] in combination with some results obtained in section 6.

- **Step 1:** We begin by establishing Property 7.5 but for functions \( \hat{v} \) belongs to \( Y(\hat{\Omega}_h) \).

Fix \( v_h \in \mathcal{Y}_h \). Let \( \hat{v} \) be the solution of the following problem

\[
\begin{align*}
\hat{v} &\in V_0(\hat{\Omega}_h) \\
\text{div} \, \hat{v} &= 0 \\
\text{curl} \, \hat{v} &= \text{curl} \, \hat{\nu}_h.
\end{align*}
\]
Lemma 7.6 For all \( p \geq 2 \), \( \hat{v} \) can be split into 
\[
\hat{v} = w + \nabla q, \quad \text{with} \quad w \in W_0^{1,p}(\hat{\Omega}_h) \quad \text{and} \quad q \in H_0^1(\hat{\Omega}_h) : \Delta q \in L^p(\hat{\Omega}_h), \tag{7.2}
\]
with estimates
\[
\|w\|_{W_0^{1,p}(\hat{\Omega}_h)} + \|\Delta q\|_{L^p(\hat{\Omega}_h)} \leq C \left( \|v_h\|_{V_h} + \|\text{curl} v_h\|_{L^p(\Omega_h)} \right). \tag{7.3}
\]

PROOF: The proof of Lemma 7.1 in [14] carries over to this case without any modification. \( \Box \)

Lemma 7.7 Let \( \hat{v} \) be the solution of problem (7.1). Then, the following estimate holds
\[
\|\hat{v} - \bar{v}_h\| \leq C \left( \|\hat{v}\|_{0,\Omega \setminus \Omega_h} + \|\hat{v} - \bar{\Pi}_N \hat{v}\|_{0,\Omega_h} \right).
\]

PROOF: Since \( \hat{v} \) is a solution of Problem (7.1), we have
\[
\|\hat{v} - \bar{v}_h\| = \|\hat{v} - \bar{v}_h\|_{0,\hat{\Omega}_h} \leq \|\hat{v}\|_{0,\Omega \setminus \Omega_h} + \|\hat{v} - v_h\|_{0,\Omega_h}.
\]
Following [14], we next use the well known Nédélec’s trick to obtain
\[
\|\hat{v} - v_h\|_{0,\Omega_h} \leq \|\hat{v} - \bar{\Pi}_N \hat{v}\|_{0,\Omega_h}.
\]
To do that, we begin by writing
\[
\|\hat{v} - v_h\|_{0,\Omega_h}^2 = (\hat{v} - v_h, \hat{v} - v_h) = (\hat{v} - v_h, \hat{v} - \bar{\Pi}_N \hat{v}) + (\hat{v} - v_h, \bar{\Pi}_N \hat{v} - v_h),
\]
and then we prove that \( (\hat{v} - v_h, \bar{\Pi}_N \hat{v} - v_h) = 0 \).

Since \( v_h \in V_h \), we have \( \bar{\Pi}_N v_h = v_h \) and we can write
\[
\bar{\Pi}_N \hat{v} - v_h = \bar{\Pi}_N (\hat{v} - \bar{v}_h).
\]
Next, we note that \( \text{curl} (\hat{v} - \bar{v}_h) = 0 \) in \( \hat{\Omega}_h \). So,
\[
\text{curl} \bar{\Pi}_N (\hat{v} - \bar{v}_h) = \bar{\Pi}_R \text{curl} (\hat{v} - \bar{v}_h) = 0.
\]
Then, there exists \( q_h \in Z_h \) such that \( \bar{\Pi}_N (\hat{v} - \bar{v}_h) = \nabla q_h \) in \( \Omega_h \). Then, from the orthogonality property of the discrete spaces \( Y_h \) and \( \nabla Z_h \), we have
\[
(v_h, \bar{\Pi}_N \hat{v} - v_h) = (v_h, \nabla q_h) = 0.
\]
Finally, since \( q_h \in H_0^1(\Omega_h) \) and \( \text{div} \hat{v} = 0 \), we get
\[
(\hat{v}, \bar{\Pi}_N \hat{v} - v_h) = (\hat{v}, \nabla q_h) = 0,
\]
and therefore the proof is complete. \( \Box \)
Lemma 7.8 Let $\hat{v}$ be the solution of problem (7.1). Then, there exists a positive constant $C$ such that

$$\|\hat{v}\|_{0,\Omega_{h}} \leq C h^{r_{h}} \|\text{curl} \hat{v}_{h}\|_{0,\hat{\Omega}_{h}}.$$

PROOF: Since $\hat{v} \in X_{N}(\hat{\Omega}_{h})$, we can use Lemma 3.7 and Corollary 3.19 in [2], with $\text{div} \hat{v} = 0$, to obtain

$$\hat{v} \in H^{r_{h}}(\hat{\Omega}_{h}),$$

$$\|\hat{v}\|_{r_{h},\hat{\Omega}_{h}} \leq C \|\hat{v}\|_{H(\text{curl}; \hat{\Omega}_{h})},$$

and

$$\|\hat{v}\|_{0,\hat{\Omega}_{h}} \leq C \|\text{curl} \hat{v}\|_{0,\hat{\Omega}_{h}}.$$

Using Lemma 6.3, the above estimates and taking into account problem (7.1), we have

$$\|\hat{v}\|_{0,\Omega \setminus \Omega_{h}} \leq C h^{r_{h}} \|\hat{v}\|_{r_{h},\Omega} \leq C h^{r_{h}} \|\hat{v}\|_{H(\text{curl}; \hat{\Omega}_{h})} \leq C h^{r_{h}} \|\text{curl} \hat{v}_{h}\|_{0,\hat{\Omega}_{h}} \leq C h^{r_{h}} \|\text{curl} \hat{v}_{h}\|_{0,\hat{\Omega}_{h}},$$

which ends the proof. $\square$

Lemma 7.9 Let $\hat{v}$ be the solution of problem (7.1). Then, there exists $\sigma > 0$ such that

$$\|\hat{v} - \hat{\Pi}_{N} \hat{v}\|_{0,\Omega_{h}} \leq C h^{\sigma} \|\bar{v}_{h}\|.$$

PROOF: The proof is identical to that of Theorem 7.4 in [14]. $\square$

• Step 2: We construct a function $v \in Y(\Omega)$ closely related to $\hat{v}$.

Let $\Psi \in H(\text{curl}; \Omega)$ be the solution of the following problem

$$\begin{cases}
-\Delta \Psi = \text{curl} \hat{v} + \hat{o} & \text{in } \Omega, \\
\text{div} \Psi = 0 & \text{in } \Omega, \\
\Psi \cdot n = 0 & \text{on } \partial \Omega, \\
\text{curl} \Psi \times n = 0 & \text{on } \partial \Omega,
\end{cases} \quad (7.4)$$

where the constant $\hat{o}$ is taken for the problem to be compatible (see Remark 6.12); i.e.,

$$\hat{o} := \frac{1}{|\Omega|} \int_{\Omega_{h} \setminus \Omega} \text{curl} \hat{v}.$$
Using Lemmas 6.1 and taking into account that $\hat{v}$ is the solution of problem (7.1), we have

$$|\hat{o}| \leq \frac{|\Omega_h \setminus \Omega|^{1/2}}{|\Omega|} \| \text{curl} \hat{v} \|_{0, \Omega_h, \Omega} \leq C h \| \text{curl} \hat{v} \|_{0, \hat{\Omega}_h} \leq C h \| \text{curl} \hat{v}_h \|_{0, \hat{\Omega}_h}. \quad (7.5)$$

Notice that, by defining $v = \text{curl} \Psi$, problem (7.4) is equivalent to problem

$$\begin{aligned}
\left\{ 
\begin{array}{l}
v \in V_0(\Omega) \\
\text{div } v = 0 \\
\text{curl } v = \text{curl } \hat{v} + \hat{o}.
\end{array}
\right.
\end{aligned} \quad (7.6)$$

Since $v \in X_N(\Omega)$, $v \in H^r(\Omega)^3$ and the results in [2] imply

$$\|v\|_{r, \Omega} \leq C \|v\|_{H(\text{curl}; \Omega)} \leq C \|\text{curl} v\|_{0, \Omega}.\)$$

So, taking into account problem (7.6) and estimate (7.5), we can obtain

$$\|v\|_{r, \Omega} \leq C \left( \|\text{curl} \hat{v}\|_{0, \Omega} + |\hat{o}| \|\Omega\|^{1/2} \right) \leq C \|\text{curl} \hat{v}_h\|_{0, \hat{\Omega}_h}.\)$$

Now, since $\Psi \in X_T(\Omega)$, we can use Lemma 3.7 and Corollary 3.16 in [2] to obtain

$$\Psi \in H^r(\Omega),$$

$$\|\Psi\|_{r, \Omega} \leq C \|\Psi\|_{H(\text{curl}; \Omega)},$$

and

$$\|\Psi\|_{0, \Omega} \leq C \|\text{curl} \Psi\|_{0, \Omega}.$$\)

Hence, because of the above estimates

$$\|\Psi\|_{r, \Omega} \leq C \|\text{curl} \Psi\|_{0, \Omega} = C \|v\|_{0, \Omega} \leq C \|\text{curl} \hat{v}_h\|_{0, \hat{\Omega}_h}. \quad (7.7)$$

According to [6], we can write

$$\Psi = w + \nabla q$$

with $w \in H^1(\Omega)^3 \cap X_T(\Omega)$ and $q \in H^{3/2+\delta}(\Omega)$, $\delta > 0$, as in splitting (6.13). Then, we can construct an extension $\Psi^e \in H^r(\mathbb{R}^3)$, with $r = 1/2 + \delta$, by defining

$$\Psi^e = w^e + \nabla q^e.$$\)

**Lemma 7.10** There exists a positive constant $C$ such that

$$\|\bar{v} - \tilde{v}\| \leq C h^{r/2} \|\bar{v}_h\|.$$
Proof: Let \( \hat{\Psi} \) be the solution of the following problem:

\[
\begin{align*}
-\Delta \hat{\Psi} &= \text{curl} \bar{v}_h \text{ in } \hat{\Omega}_h, \\
\text{div} \hat{\Psi} &= 0 \quad \text{in } \hat{\Omega}_h, \\
\hat{\Psi} \cdot n &= 0 \quad \text{on } \partial \hat{\Omega}_h, \\
\text{curl} \hat{\Psi} \times n &= 0 \quad \text{on } \partial \hat{\Omega}_h,
\end{align*}
\]

(7.8)

then, \( \hat{v} = \text{curl} \hat{\Psi} \) is a solution of problem (7.1). Since \( \hat{\Psi} \in X_T(\hat{\Omega}_h) \), \( \hat{\Psi} \in H^{r_h}(\hat{\Omega}_h) \) and

\[
\| \hat{\Psi} \|_{r_h, \hat{\Omega}_h} \leq C \| \hat{\Psi} \|_{H(\text{curl}; \hat{\Omega}_h)} \leq C \| \text{curl} \hat{\Psi} \|_{0, \hat{\Omega}_h} = C \| \hat{v} \|_{0, \hat{\Omega}_h}
\]

(7.9)

Let us consider an extension \( \Psi^e \) of \( \Psi \) as above. Then, integrating by parts and using (7.1) and (7.6), we obtain

\[
\| \hat{v} - v \|_{0, \Omega}^2 = \int_\Omega \text{curl} (\hat{\Psi} - \Psi) \cdot (\hat{v} - v) = \int_{\hat{\Omega}_h} \text{curl} (\hat{\Psi} - \Psi^e) \cdot (\hat{v} - v) \\
- \int_{\Omega_h \setminus \hat{\Omega}_h} \text{curl} (\hat{\Psi} - \Psi^e) \cdot \hat{v} = \int_{\hat{\Omega}_h} (\hat{\Psi} - \Psi^e) \cdot \text{curl} (\hat{v} - v) \\
- \int_{\Omega_h \setminus \hat{\Omega}_h} \text{curl} (\hat{\Psi} - \Psi^e) \cdot \hat{v} = \int_\Omega (\hat{\Psi} - \Psi) \cdot \hat{o} + \int_{\hat{\Omega}_h \setminus \Omega} (\hat{\Psi} - \Psi^e) \cdot \text{curl} \hat{v} \\
- \int_{\Omega_h \setminus \hat{\Omega}_h} \text{curl} (\hat{\Psi} - \Psi^e) \cdot \hat{v}.
\]

Now, we are going to estimate the terms in the right hand side of the inequality above.

The first term is easily bounded by using the Cauchy-Schwarz inequality, estimates (7.5), (7.7) and (7.9):

\[
\left| \int_\Omega (\hat{\Psi} - \Psi) \cdot \hat{o} \right| \leq (\| \hat{\Psi} \|_{0, \Omega} + \| \Psi \|_{0, \Omega}) \| \hat{o} \|_{\Omega}^{1/2} \leq C h \| \text{curl} \hat{v}_h \|_{0, \hat{\Omega}_h}^2.
\]

For the second one, we use Cauchy-Schwarz inequality, Lemma 6.3, (6.15),
(7.7) and (7.9) to obtain

\[ \left| \int_{\Omega_h \setminus \Omega} (\hat{\Psi} - \Psi^e) \cdot \text{curl} \  \hat{\nu} \right| \leq \| \hat{\Psi} - \Psi^e \|_{0, \Omega_h \setminus \Omega} \| \text{curl} \  \hat{\nu} \|_{0, \Omega_h \setminus \Omega} \]

\[ \leq \left( \| \hat{\Psi} \|_{0, \Omega_h \setminus \Omega} + \| \Psi^e \|_{0, \Omega_h \setminus \Omega} \right) \| \text{curl} \  \hat{\nu} \|_{0, \hat{\Omega}_h} \]

\[ \leq C \left( h^{r_h} \| \hat{\Psi} \|_{r_h, \Omega_h} + h^r \| \Psi^e \|_{r, \Omega_h} \right) \| \text{curl} \  \hat{\nu} \|_{0, \hat{\Omega}_h} \]

\[ \leq C h^{r_h} \left( \| \hat{\Psi} \|_{r_h, \Omega_h} + \| \Psi \|_{H(\text{curl}; \Omega)} \right) \| \text{curl} \  \hat{\nu} \|_{0, \hat{\Omega}_h} \]

\[ \leq C h^{r_h} \| \text{curl} \  \bar{\nu}_h \|_{0, \hat{\Omega}_h}^2 , \]

with \( \hat{r}_h = \min\{r, r_h\} \).

For the last term, we use Cauchy-Schwarz inequality, Lemma 7.8, estimates (6.17), (7.7) and (7.9) and we get

\[ \left| \int_{\Omega_h \setminus \Omega} \text{curl} \ (\hat{\Psi} - \Psi^e) \cdot \hat{\nu} \right| \leq \| \text{curl} \ (\hat{\Psi} - \Psi^e) \|_{0, \Omega_h \setminus \Omega} \| \hat{\nu} \|_{0, \Omega_h \setminus \Omega} \]

\[ \leq C h^{r_h} \left( \| \text{curl} \  \hat{\nu} \|_{0, \Omega_h \setminus \Omega} + \| \Psi \|_{0, \Omega_h \setminus \Omega} \right) \| \text{curl} \  \hat{\nu} \|_{0, \hat{\Omega}_h} \]

\[ \leq C h^{r_h} \| \text{curl} \  \bar{\nu}_h \|_{0, \hat{\Omega}_h}^2 . \]

Thus, combining the above inequalities we conclude the proof. \( \square \)

**Step 3**: We end by establishing property (7.5).

For any \( \nu_h \in \mathcal{Y}_h \), we can write

\[ \| \bar{\nu} - \bar{\nu}_h \| \leq \| \bar{\nu} - \tilde{\nu} \| + \| \tilde{\nu} - \bar{\nu}_h \| , \]

with \( \tilde{\nu} \) and \( \bar{\nu} \) as obtained in steps 1 and 2, respectively. Then, by using Lemmas 7.7, 7.8, 7.9 and 7.10 we have

\[ \| \bar{\nu} - \bar{\nu}_h \| \leq C \left( h^{r_h} + h^r + h^{r_h}/2 \right) \| \bar{\nu}_h \| . \]

Therefore, (7.5) holds.

### 8 Approximation of the eigenfunctions

Let \( f \in \mathbf{G} \). Define \( \bar{u} := A f \) and \( \bar{u}_h := A_h \bar{f} \).
Lemma 8.1 There exists a positive constant $C$ such that

$$
\| \tilde{u} - \tilde{u}_h \| \leq \left( \| u - \tilde{u} \|_{H(\text{curl}; \Omega \cap \Omega_h)} + \inf_{v_h \in V_h} \| \tilde{u} - v_h \|_{V_h} \\
+ \sup_{w_h \in V_h} \frac{|a_h(\tilde{u} - u_h, w_h)|}{\| w_h \|_{V_h}} \right) + \| \tilde{u} \|_{H(\text{curl}; \Omega \cap \Omega_h)} + \| \hat{u} \|_{H(\text{curl}; \Omega \cap \Omega_h)}.
$$

Proof: We have

$$
\| \tilde{u} - \tilde{u}_h \|^2 = \| \tilde{u} - \tilde{u}_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} \\
\leq \| u - u_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} + \| u \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} + \| u_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)}.
$$

Now, let $v_h$ be an arbitrary element in the space $V_h$. We can write

$$
\| u - u_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} \leq 2 \left( \| u - v_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} + \| v_h - u_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} \right)
$$

and

$$
\| u_h \|_{H(\text{curl}; \Omega \cap \Omega_h)} \leq \| u_h - v_h \|_{H(\text{curl}; \Omega \cap \Omega_h)} + \| v_h \|_{H(\text{curl}; \Omega \cap \Omega_h)}.
$$

By using the uniform coerciveness and continuity of the bilinear form $a_h$, we obtain

$$
\alpha \| v_h - u_h \|^2_{H(\text{curl}; \Omega \cap \Omega_h)} \\
\leq \int_{\Omega_h} \mu^{-1} \text{curl}(v_h - u_h) \cdot \text{curl}(v_h - u_h) + \int_{\Omega_h} \varepsilon(v_h - u_h) \cdot (v_h - u_h) \\
= \int_{\Omega_h} \mu^{-1} \text{curl}(\hat{u} - u_h) \cdot \text{curl}(v_h - u_h) + \int_{\Omega_h} \varepsilon(\hat{u} - u_h) \cdot (v_h - u_h) \\
+ \int_{\Omega_h} \mu^{-1} \text{curl}(\tilde{u} - u_h) \cdot \text{curl}(v_h - u_h) + \int_{\Omega_h} \varepsilon(\tilde{u} - u_h) \cdot (v_h - u_h) \\
\leq \| v_h - \hat{u} \|_{V_h} \| v_h - u_h \|_{V_h} + a_h(\hat{u} - u_h, v_h - u_h),
$$

from which we deduce

$$
\| v_h - u_h \|_{V_h} \leq C \left( \| v_h - \hat{u} \|_{V_h} + \sup_{w_h \in V_h} \frac{a_h(\hat{u} - u_h, w_h)}{\| w_h \|_{V_h}} \right).
$$

On the other hand,

$$
\| u - v_h \|_{H(\text{curl}; \Omega \cap \Omega_h)} \leq \| u - \tilde{u} \|_{H(\text{curl}; \Omega \cap \Omega_h)} + \| \tilde{u} - v_h \|_{H(\text{curl}; \Omega \cap \Omega_h)}
$$

and

$$
\| v_h \|_{H(\text{curl}; \Omega \cap \Omega_h)} \leq \| v_h - \hat{u} \|_{H(\text{curl}; \Omega \cap \Omega)} + \| \hat{u} \|_{H(\text{curl}; \Omega \cap \Omega)}.
$$

Combining the above inequalities, we conclude the proof. □
We are going to estimate the terms appearing in the right hand side of the inequality in the previous Lemma. Observe that the first term is directly bounded by Lemma 6.11.

**Lemma 8.2** There exists a positive constant $C$ such that

$$\inf_{v_h \in V_h} \|\hat{u} - v_h\|_{V_h} \leq C h^{r_h} \|\bar{f}\|_{0,\mathbb{R}^3}.$$  

**Proof:** Since $\hat{u} \in H^{r_h}(\text{curl};\hat{\Omega}_h)$, the interpolations operators $\Pi_N$ and $\widehat{\Pi}_N$ are well defined. Then, using equation (6.2) and Lemma 6.5, we obtain

$$\|\hat{u} - \widehat{\Pi}_N \hat{u}\|_{V_h} \leq \|\hat{u} - \Pi_N \hat{u}\|_{V_h} + \|\Pi_N \hat{u} - \widehat{\Pi}_N \hat{u}\|_{V_h} \leq C h^{r_h} \|\hat{u}\|_{H^{r_h}(\text{curl};\Omega_h)}.$$  

Choosing $v_h = \widehat{\Pi}_N \hat{u}$, equation (6.28) yields

$$\|\hat{u} - v_h\|_{V_h} \leq C h^{r_h} \|\bar{f}\|_{0,\mathbb{R}^3}.$$  

So, taking the infimum with respect to $v_h \in V_h$, we can conclude the proof. \qed

**Lemma 8.3** There exists a positive constant $C$ such that

$$\sup_{w_h \in V_h} \frac{|a_h(\hat{u} - u_h, w_h)|}{\|w_h\|_{V_h}} \leq C h^{r_h} \|\bar{f}\|_{0,\mathbb{R}^3}.$$  

**Proof:** Let $w_h \in G_h$.

\begin{align*}
    a_h(\hat{u} - u_h, w_h) &= \int_{\Omega_h} \mu^{-1} \text{curl}(\hat{u} - u_h) \cdot \text{curl} w_h + \int_{\Omega_h} \varepsilon(\hat{u} - u_h) \cdot w_h \\
    &= \int_{\hat{\Omega}_h} \mu^{-1} \text{curl} \hat{u} \cdot \text{curl} \bar{w}_h + \int_{\hat{\Omega}_h} \varepsilon \hat{u} \cdot \bar{w}_h \\
    &\quad - \int_{\Omega_h} \mu^{-1} \text{curl} u_h \cdot \text{curl} w_h - \int_{\Omega_h} \varepsilon u_h \cdot w_h \\
    &= \int_{\hat{\Omega}_h} \mu^{-1} \text{curl} \hat{u} \cdot \text{curl} \bar{w}_h + \int_{\hat{\Omega}_h} \varepsilon \hat{u} \cdot \bar{w}_h - \int_{\Omega_h} \varepsilon \bar{f} \cdot w_h \\
    &= \int_{\hat{\Omega}_h} \mu^{-1} \text{curl} \hat{u} \cdot \text{curl} \bar{w}_h + \int_{\hat{\Omega}_h} \varepsilon u^\varepsilon \cdot \bar{w}_h \\
    &\quad + \int_{\Omega_h} (\varepsilon \hat{u} - \varepsilon u^\varepsilon) \cdot \bar{w}_h - \int_{\Omega_h} \varepsilon \bar{f} \cdot w_h.
\end{align*}
Now, consider the decomposition

\[ \tilde{w}_h = \nabla \xi + \varepsilon^{-1} \text{curl} \varphi \]

with \( \xi \in H^1_0(\hat{\Omega}_h) \) and \( \varphi \in H(\text{div}^0; \hat{\Omega}_h) \) as in Lemma 6.8. Then, integrating by parts, we obtain

\[
\int_{\hat{\Omega}_h} \mu^{-1} \text{curl} \hat{u} \cdot \text{curl} \tilde{w}_h + \int_{\hat{\Omega}_h} \varepsilon u^e \cdot \tilde{w}_h = \int_{\hat{\Omega}_h} \mu^{-1} (\text{curl} u)^e + (\mu \varepsilon)^{-1} \nabla \varepsilon \times (u^e - \hat{u}) \cdot \text{curl} \tilde{w}_h + \int_{\hat{\Omega}_h} \mu \varepsilon^e \cdot \tilde{w}_h
\]

Inserting (8.2) into (8.1) results in

\[
a_h(\hat{u} - u_h, w_h) = -\int_{\hat{\Omega}_h} \varepsilon f \cdot \nabla \xi + \int_{\hat{\Omega}_h} \varepsilon u^e \cdot \nabla \xi
\]

\[
+ \int_{\hat{\Omega}_h} (\mu \varepsilon)^{-1} \nabla \varepsilon \times (u^e - \hat{u}) \cdot \text{curl} \tilde{w}_h + \int_{\hat{\Omega}_h} (\mu \varepsilon)^{-1} \hat{c} \cdot \text{curl} \tilde{w}_h
\]

\[
+ \int_{\hat{\Omega}_h} (\mu \varepsilon)^{-1} \nabla \varepsilon \times (u^e - \hat{u}) \cdot \text{curl} \tilde{w}_h + \int_{\hat{\Omega}_h} (\mu \varepsilon)^{-1} \hat{c} \cdot \text{curl} \tilde{w}_h.
\]
We are going to estimate the terms in the right hand side above.

Using the Cauchy-Schwarz inequality and Lemma 6.8, we obtain for the first term
\[
\left| \int_{\Omega} \varepsilon f \cdot \nabla \xi \right| \leq \| \varepsilon^{1/2} f \|_{0,\Omega} \| \varepsilon^{1/2} \nabla \xi \|_{0,\Omega} \leq C h^{r_h} \| f \|_{0,\Omega} \| W_h \|_{V_h}.
\]

The Cauchy-Schwarz inequality, Lemma 6.8, estimates (6.19) and (6.24) yield
\[
\left| \int_{\Omega} \varepsilon u^e \cdot \nabla \xi \right| \leq \| \varepsilon^{1/2} u^e \|_{0,\Omega} \| \varepsilon^{1/2} \nabla \xi \|_{0,\Omega} \leq C h^{r_h} \| u^e \|_{0,\Omega} \| W_h \|_{V_h}
\]
\[
\leq C h^{r_h} \| u \|_{r,\Omega} \| W_h \|_{V_h} \leq C h^{r_h} \| f \|_{0,\Omega} \| W_h \|_{V_h}.
\]

Again, using the Cauchy-Schwarz inequality, Lemma 6.8, estimates (6.19), (6.20) and (6.21), we obtain for the third term
\[
\left| \int_{\Omega} \mu^{-1} \varepsilon (\nabla u^e) + \varepsilon u^e \cdot \varepsilon^{-1} \nabla \varphi \right|_{\Omega_h \setminus \Omega}
\leq \| \mu^{-1} \nabla (\nabla u^e) + \varepsilon u^e \|_{0,\Omega} \| \varepsilon^{-1} \nabla \varphi \|_{0,\Omega} \Omega
\leq C h^{r} \left( \| \nabla (\nabla u^e) \|_{0,\Omega} + \| \varepsilon u^e \|_{0,\Omega} \right) \| W_h \|_{V_h}
\leq C h^{r_h} \| f \|_{0,\Omega} \| W_h \|_{V_h}.
\]

Finally, using the Cauchy-Schwarz inequality, Lemmas 6.11 and 6.3, estimates (6.19) and (6.28), we obtain
\[
\left| \int_{\Omega_h} (\varepsilon \hat{u} - \varepsilon u^e) \cdot \bar{w}_h \right| \leq \left| \int_{\Omega_h} (\varepsilon \hat{u} - \varepsilon u^e) \cdot \bar{w}_h \right| + \left| \int_{\Omega_h} (\varepsilon \hat{u} - \varepsilon u^e) \cdot \bar{w}_h \right|
\leq \left( \| \varepsilon (\hat{u} - u) \|_{0,\Omega \setminus \Omega_h} + \| \varepsilon u^e \|_{0,\Omega \setminus \Omega_h} \right) \| W_h \|_{V_h}
\leq C \left( h^{r_h} \| f \|_{0,\Omega} + h^{r_h} \| u^e \|_{r,\Omega} + h^{r} \| u^e \|_{r,\Omega} \right) \| W_h \|_{V_h}
\leq h^{r_h} \| f \|_{0,\Omega} \| W_h \|_{V_h}.
\]

\[
\left| \int_{\Omega_h} (\mu \varepsilon)^{-1} \nabla \varepsilon \times (u^e - \hat{u}) \cdot \nabla \bar{w}_h \right| \leq \left| \int_{\Omega_h} (\mu \varepsilon)^{-1} \nabla \varepsilon \times (u - \hat{u}) \cdot \nabla \bar{w}_h \right|
\]

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Therefore, by replacing this result in equation \((8.5)\) we have
\[
a_h(\hat{u} - u_h, w_h) = \int_{\Omega_h} \varepsilon(\hat{u} - u_h) \cdot \tilde{w}_h + \int_{\Omega_h \setminus \Omega} \varepsilon u^e \cdot w_h + \int_{\Omega} \varepsilon f \cdot \tilde{w} - \int_{\Omega} \varepsilon f \cdot w_h. \tag{8.6}
\]

The first two terms can be easily bounded. In fact, by using the Cauchy-Schwartz inequality
\[
\left| \int_{\Omega_h} \varepsilon(\hat{u} - u^e) \cdot \tilde{w}_h \right| \leq \left( \|\varepsilon(\hat{u} - u)\|_{0,\Omega \cap \Omega_h} + \|\varepsilon(\hat{u} - u^e)\|_{0,\Omega_h \setminus \Omega} \right) \|w_h\|_{0,\Omega_h} \tag{8.7}
\]
\[
\leq C \left( h^{r_h} \|f\|_{0,\Omega} + h^{r_h} \|\varepsilon \hat{u}\|_{r_h,\Omega_h} + h^{r} \|u^e\|_{r,\Omega_h} \right) \|w_h\|_{\Omega_h}
\]
\[
\leq C h^{r_h} \|f\|_{0,\Omega} \|w_h\|_{\Omega_h}.
\]

So, combining the above bounds gives
\[
\sup_{w_h \in G_h} \frac{|a_h(\hat{u} - u_h, w_h)|}{\|w_h\|_{\Omega_h}} \leq C h^{r_h} \|f\|_{0,\Omega}. \tag{8.4}
\]
and

\[
\int_{\Omega_h \setminus \Omega} \varepsilon \mathbf{u}^r \cdot \mathbf{w}_h \leq \|\varepsilon \mathbf{u}^r\|_{0, \Omega_h \setminus \Omega} \|\mathbf{w}_h\|_{0, \Omega_h \setminus \Omega} \leq C h^r \|\mathbf{u}^e\|_{0, \Omega_h} \|\mathbf{w}_h\|_{V_h}. \tag{8.8}
\]

To bound the last two terms, we observe that

\[
\int_{\Omega} \varepsilon f \cdot \hat{w} - \int_{\Omega_h} \varepsilon \bar{f} \cdot w_h = \sum_{T \in T_h} \int_{\mathcal{T} \cap \Gamma \neq \emptyset} \int_{\partial T} \varepsilon \cdot \hat{w} - \sum_{T \in T_h} \int_{\partial T \cap \Gamma \neq \emptyset} \varepsilon \cdot w_h.
\]

Denoting by \(\omega_T\) the domain defined by \(T \setminus T_{id}\), we have that the sums in the right hand side can be written as follows

\[
\sum_{T \in T_{id}^d} \int_{\partial T \cap \Gamma \neq \emptyset} \varepsilon \cdot (\hat{w} - \bar{w}_h) = \sum_{T \in T_h} \sum_{T \in T_{id}^d} \int_{\partial T \cap \Gamma \neq \emptyset} \varepsilon \cdot w_h = \sum_{T \in T_{id}^d} \int_{\omega_T} \varepsilon \cdot \nabla(\hat{p} - \bar{p}_h).
\]

Now, using Lemma 6.10, we obtain

\[
\|\hat{p} - p_h\|_{1, T_{id}^d} \leq C \begin{cases} h \|p_h\|_{1,T} & \text{if } T_{id} \subset T, \\ 0, & \text{otherwise} \end{cases}.
\]

Hence, we obtain

\[
\left| \int_{\Omega} \varepsilon f \cdot \hat{w} - \int_{\Omega_h} \varepsilon \bar{f} \cdot w_h \right| \leq C h^r \|f\|_{0, \Omega} \|w_h\|_{V_h}. \tag{8.9}
\]

Inserting (8.7), (8.8) and (8.9) into equation (8.6), we obtain

\[
\sup_{w_h \in K_h} \frac{|a_h(\mathbf{u} - \mathbf{u}_h, w_h)|}{\|w_h\|_{V_h}} \leq C h^r \|f\|_{0, \Omega}. \tag{8.10}
\]

So, from inequalities (8.4) and (8.10) we conclude the lemma. \(\square\)

**Lemma 8.4** For all \(f \in G\), there exists a positive constant \(C\) such that

\[
\|(A - A_h)\bar{f}\| \leq C h^r \|f\|_{0, \Omega}. \tag{8.11}
\]

**Proof:** It is an immediate consequence of the previous lemmas. \(\square\)
Lemma 8.5 For all $f_h^+ \in K_h$, there exists a positive constant $C$ such that

$$\|(A - A_h)\tilde{f}_h^+\| \leq C h^{1/2}\|f_h^+\|_{0,\Omega_h}.$$  \hfill (8.12)

PROOF: Let us define $\tilde{u} := A\tilde{f}_h^+$ and $\tilde{u}_h := A_h\tilde{f}_h^+$. We prove first that $u_h = f_h^+$. Since $f_h^+ \in K_h$, $f_h^+ \in H_0(\text{curl}\Omega, \Omega_h)$ and it is straightforward to show that

$$a_h(f_h^+, v_h) = \int_{\Omega_h} \varepsilon \nabla f_h^+ \cdot v_h = b_h(f_h^+, v_h), \quad \forall v_h \in V_h.$$

Then, $u_h = f_h^+$ is a solution and, because problem (5.1) is well posed, this solution is unique. As a consequence, $A_h\tilde{f}_h^+ = \tilde{f}_h^+$.

Now, since $f_h^+ \in K_h$, we may write $f_h^+ = \nabla p_h$, with $p_h \in Z_h$. Let $\hat{p}$ be the function associated with $p_h$ as in Definition 6.9. Since $\hat{p} \in H_0^1(\Omega)$, then $f := \nabla \hat{p} \in H_0(\text{curl}\Omega, \Omega) = K$. Therefore, by using the triangle inequality,

$$\|(A - A_h)\tilde{f}_h^+\| = \|\tilde{u} - \tilde{f}_h^+\| \leq \|\tilde{u} - \hat{f}\| + \|\hat{f} - \tilde{f}_h^+\|.$$  \hfill (8.13)

Next, we are going to bound the first term on the right hand side of the inequality above. Let $v$ be an arbitrary element in $V_0(\Omega)$. From the definition of $a$ and problem (3.4), we have

$$a(\hat{f} - u, v) = \int_{\Omega} \varepsilon \nabla \hat{p} \cdot v - \int\Omega_h \varepsilon \nabla \hat{p} \cdot v = \int\Omega_\Omega_h \varepsilon \nabla \hat{p} \cdot v - \int\Omega_\Omega_h \varepsilon f_h^+ \cdot v$$

$$= \sum_{T \in T_h} \int_{T} \varepsilon \nabla p_h \cdot v \bigg|_{T} + \sum_{T \cap \Gamma \neq \emptyset} \int_{T} \varepsilon \nabla \hat{p} \cdot v - \int_{T} \varepsilon f_h^+ \cdot v \bigg|_{T \subset T}$$

$$= \int_{\Omega_\Omega_h} \varepsilon \nabla p_h \cdot v + \sum_{T \cap \Gamma \neq \emptyset} \int_{T} \varepsilon \nabla (\hat{p} - p_h) \cdot v - \int_{\Omega_\Omega_h} \varepsilon f_h^+ \cdot v \bigg|_{T \subset T}$$

$$= \sum_{T \cap \Gamma \neq \emptyset} \int_{T} \varepsilon \nabla (\hat{p} - p_h) \cdot v.$$
So, by choosing $v = \hat{f} - u$, we can obtain
\[
\|\bar{\hat{f}} - \bar{u}\| \leq C \sum_{\mathcal{T}_{id} \in \mathcal{T}_h} \left( \|\nabla (\hat{p} - p_h)\|_{0,\mathcal{T}_{id}}^2 \right)^{1/2}
\]
(8.14)
as a direct consequence of the coerciveness of $a$.

On the other hand,
\[
\|\bar{\hat{f}} - \bar{f}_h\|^2 = \int_{\Omega \cup \Omega_h} \nabla (\hat{p} - p_h) \cdot \nabla (\hat{p} - p_h) = \sum_{\mathcal{T}_{id} \in \mathcal{T}_h} \int_{\mathcal{T}_{id}} \nabla (\hat{p} - p_h) \cdot \nabla (\hat{p} - p_h)
\]
\[
+ \int_{\omega_T} \nabla p_h \cdot \nabla p_h.
\]
(8.15)

Now, taking into account Lemma 6.1, we can obtain
\[
\|\nabla p_h\|_{0,\omega_T} = |\nabla p_h|^{1/2} \leq C h^{1/2} \|\nabla p_h\|_{0,T}.
\]
(8.16)

Then, by combining (8.13) with estimates (8.14), (8.15) and (8.16) and using Lemma (6.10), we have
\[
\|\bar{\hat{u}} - \bar{u}_h\| \leq C h^{1/2}(1 + h^{1/2})\|\nabla p_h\|_{0,\Omega_h} = C h^{1/2}\|f_h\|_{0,\Omega_h}.
\]

Thus, we conclude the proof. □

**Theorem 8.6** (P1) *There exists $\eta_h$, tending to zero as $h$ goes to zero, such that*
\[
\left\| (A - A_h)\bar{f}_h \right\| \leq \eta_h \left\| \bar{f}_h \right\| \quad \forall \bar{f}_h \in \tilde{V}_h(\mathbb{R}^3).
\]

**Proof:** We begin by writing the orthogonal decomposition
\[
f_h = f_h^\parallel \oplus f_h^\perp,
\]
with $f_h^\parallel \in G_h$ and $f_h^\perp \in K_h$. Let $f \in G$. Then
\[
\left\| (A - A_h)\bar{f}_h \right\| \leq \left\| (A - A_h)\bar{f}_h^\parallel \right\| + \left\| (A - A_h)\bar{f}_h^\perp \right\|
\leq \left\| (A - A_h)(\bar{f} - \bar{f}_h) \right\| + \left\| (A - A_h)\bar{f} \right\| + \left\| (A - A_h)\bar{f}_h^\perp \right\|.
\]
(8.17)
Notice that the third term in the right hand side of (8.17) is directly bounded by Lemma 8.5. To deal with the first term, we denote

\[
\|A\|_{L(X(\mathbb{R}^3), \tilde{V}(\mathbb{R}^3))} = \sup_{v \in X(\mathbb{R}^3)} \|Av\|,
\]

\[
\|A_h\|_{L(X(\mathbb{R}^3), \tilde{V}_h(\mathbb{R}^3))} = \sup_{v \in X(\mathbb{R}^3)} \|A_hv\|.
\]

Since \( A \) and \( A_h \) are bounded (and \( A_h \), uniformly on \( h \)), we have

\[
\|(A - A_h)(\bar{f} - \bar{f}_h)\| \leq \left( \|A\|_{L(X(\mathbb{R}^3), \tilde{V}(\mathbb{R}^3))} + \|A_h\|_{L(X(\mathbb{R}^3), \tilde{V}_h(\mathbb{R}^3))} \right) \|\bar{f} - \bar{f}_h\|_{0, \mathbb{R}^3} \leq C \eta_h \|\tilde{f}\|_V, \]

the latter because of Property (7.2).

The second term can be bounded by using Lemma 8.4 as follows:

\[
\|(A - A_h)\bar{f}\| \leq C h^{\bar{r}_h} \|\bar{f}\|_{0, \mathbb{R}^3} \leq C h^{\bar{r}_h} (\|\bar{f} - \bar{f}_h\|_{0, \mathbb{R}^3} + \|\bar{f}_h\|_{0, \mathbb{R}^3}) \leq C h^{\bar{r}_h} (\eta_h + 1) \|\tilde{f}\|_V, \]

where we have used again Property (7.2). So, we conclude the proof. \( \square \)

**Theorem 8.7 (P2)** For each eigenfunction \( \bar{u} \) of \( A \) associated to \( \lambda \), there exists a strictly positive constant \( C \) such that

\[
\|\bar{u}\|_{H(\text{curl}; \Omega \setminus \Omega_h)} = \|u\|_{H(\text{curl}; \Omega \setminus \Omega_h)} \leq C h^{r} \|u\|_{H^r(\text{curl}; \Omega)}. \]

**Proof:** It is an immediate consequence of the regularity of the eigenfunctions \( u \) of the operator \( T \) and Lemma 6.3. \( \square \)

**Theorem 8.8 (P3)** For each eigenfunction \( \bar{v} \) of \( A \) associated to \( \lambda \), there exists a strictly positive constant \( C \) such that

\[
\inf_{v_h \in V_h} \|\bar{u} - v_h\| \leq C h^{\bar{r}_h} \|u\|_{H^r(\text{curl}; \Omega)}. \]

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Proof: Considering \( \hat{u} \) as in Lemma 6.11, we can use the triangle inequality to obtain
\[
\| \hat{u} - \bar{v}_h \|^2 = \| u \|^2_{H(\text{curl}; \Omega \setminus \Omega_h)} + \| \bar{u} - v_h \|^2_{V_h}
\leq \| u \|^2_{H(\text{curl}; \Omega \setminus \Omega_h)} + \| \hat{u} - \bar{u} \|^2_{V_h} + \| \bar{u} - v_h \|^2_{V_h}
\leq \| u \|^2_{H(\text{curl}; \Omega \setminus \Omega_h)} + \| u - \hat{u} \|^2_{H(\text{curl}; \Omega \cap \Omega_h)}
\quad + \| \hat{u} \|^2_{H(\text{curl}; \Omega \setminus \Omega)} + \| \hat{u} - v_h \|^2_{V_h}.
\]

Now, from Lemma 6.11, we have
\[
\| u - \hat{u} \|_{H(\text{curl}; \Omega \cap \Omega_h)} \leq C h^r \| u \|_{0, \Omega}.
\]
On the other hand, because of the regularity results (3.6) and (6.28)
\[
\| u \|_{H(\text{curl}; \Omega \setminus \Omega_h)} \leq C h^r \| u \|_{r(\text{curl}; \Omega)}
\text{and}
\| \hat{u} \|_{H(\text{curl}; \Omega \setminus \Omega)} \leq C h^r \| \hat{u} \|_{r(\text{curl}; \Omega_h)} \leq C h^r \| u \|_{0, \Omega}.
\]
Finally, from Lemma 8.2, we have
\[
\inf_{v_h \in V_h} \| \hat{u} - v_h \|_{V_h} \leq C h^r \| u \|_{0, \Omega}.
\]
Then combining the previous inequalities, we conclude the proof. □

Theorem 8.9 (P4) There exists a positive constant \( C \) such that
\[
\| (A - A_h)\hat{u} \| \leq C h^r \| \hat{u} \| \quad \forall u \in \mathbf{E}(\mathbb{V}(\mathbb{R}^3)).
\]

Proof: It is a direct consequence of Lemma 8.4. □

By virtue of the previous theorems, the spectrum of \( A_h \) furnishes the approximations of the spectrum of \( A \) as we stated in section 5.

9 Approximation of the eigenvalues

In order to prove a double order error estimate for the approximate eigenvalues, we have to estimate the consistency terms appearing in Theorem 5.6.

Lemma 9.1 There exists a positive constant \( C \) such that
\[
M_h = \sup_{x \in \mathbf{E}(\mathbb{V}(\mathbb{R}^3))} \sup_{y \in \mathbf{E}(\mathbb{V}(\mathbb{R}^3))} | a_h(Ax, \Pi_h y - y) - b_h(x, \Pi_h y - y) | \leq C h^{2r},
\]
where \( \| x \| = 1 \quad \| y \| = 1 \).
with \( \Pi_h \) being the projection onto \( \tilde{V}_h(\mathbb{R}^3) \) with respect to \( a_h \), defined by equation (5.3).

**Proof:** Let \( x \in E(V(\mathbb{R}^3)) \), with \( \|x\|_{H(\text{curl};\Omega)} = 1 \), and put \( w = \tilde{S} A x \). According to the definition of \( A \), we have \( w = \tilde{S} \tilde{T} \tilde{S} x = \tilde{T} \tilde{S} x \). From (3.6) we know that \( w \in H^r(\text{curl};\Omega) \) and

\[
\|w\|_{H^r(\text{curl};\Omega)} \leq C \|x\|_{H(\text{curl};\Omega)} = C.
\]

Now, taking \( v \in D(\Omega) \) as test functions in (3.4), it can be shown that \( w \) is the solution of the following strong problem

\[
\begin{aligned}
\text{curl} \left( \mu^{-1} \text{curl} w \right) + \varepsilon w &= \varepsilon x, \quad \text{in } \Omega, \\
w \times n &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
\]

Let us denote by \( \bar{w} \) the extension of \( w \) by zero from \( \Omega \) to \( \mathbb{R}^3 \). Let \( y \in E(V(\mathbb{R}^3)) \), with \( \|y\|_{H(\text{curl};\Omega)} = 1 \), and take \( v_h = \Pi_h y - y \in H_0(\text{curl};\Omega_h) \). Integrating by parts, we obtain

\[
|a_h(\bar{w}, v_h) - b_h(x, v_h)|
\]

\[
= \left| - \int_{\Omega_h} \mu^{-1} \text{curl} w \cdot \text{curl} v_h - \int_{\Omega_h} \varepsilon w \cdot v_h + \int_{\Omega_h} \varepsilon x \cdot v_h \right|
\]

\[
\leq C \left( \|\text{curl} w\|_{0,\Omega;\Omega_h} \|\text{curl} v_h\|_{0,\Omega;\Omega_h} + \|w\|_{0,\Omega;\Omega_h} \|v_h\|_{0,\Omega;\Omega_h}
\right.
\]

\[
+ \|x\|_{0,\Omega;\Omega_h} \|v_h\|_{r,\Omega;\Omega_h} + \|w\|_{r,\Omega} \|y\|_{r,\Omega} + \|x\|_{r,\Omega} \|y\|_{r,\Omega}\).
\]

In the last inequality, we have used that \( v_h|_{\Omega;\Omega_h} = -y|_{\Omega;\Omega_h} \) and the estimate in Lemma 6.3. Finally, we can conclude the proof using estimate (3.6).

**Lemma 9.2** There exists a positive constant \( C \) such that

\[
N_h = \sup_{x \in E(V(\mathbb{R}^3))} \sup_{y \in E(V(\mathbb{R}^3))} |a_h(Ax, y) - b_h(x, y)| \leq C h^{2r}.
\]

**Proof:** It is identical to that of the previous lemma by substituting \( v_h \) by \( y \).

As a conclusion of the previous lemma, we obtain optimal order error estimates for the approximate eigenvalues.
Theorem 9.3 There exists a positive constant $C$ such that
\[
\max_{i=1,\ldots,m} |\lambda - \lambda_{ih}| \leq Ch^{2^k_h}.
\]

Proof: It is a direct consequence of properties P2, P3, P4 and the previous lemmas. □

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References


