Clasificación vía Teoría Descriptiva de Conjuntos

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Example 4: (Kolmogorov-Sinai 1955, Ornstein 1971) Bernoulli shifts, the entropy is a complete invariant.
Example 5: (Halmos-von Neumann 1939) Conjugacy of ergodic m.p. transformations with discrete spectrum. The spectrum is a complete invariant.

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Problem: What about torsion free abelian groups of higher rank?

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Idea: use the unsolvability of the word problem for certain groups. Make this group the fundamental group of the manifold.
The underlying notion of non classification comes from recursion theory from mathematical logic.

Classification meta mathematically speaking

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In what follows we try to convey why descriptive set theory provides an adequate framework to study many classification problems.

Classical descriptive set theory

Recall that a *Polish space* is a completely metrizable separable topological space. E.g. $\mathbb{R}$, $\mathbb{N}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$, $C([0,1])$.

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- *Borel sets and functions*, i.e. those sets that occur in the σ-algebra generated by the open sets.
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Definition. A *standard Borel space* is a set X equipped with a σ-algebra $\mathcal{B}$ which is itself generated by the open sets of a Polish topology on X.

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This set is seen to be G_δ and thus is Polish.

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To give a Borel structure to $\mathrm{vN}(\mathcal{H})$ was the first step in Effros's attempt to show that there exists uncountably many factors.

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Analytic equivalence relations

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Theorem: Foreman-Rudolph-Weiss 2006-2010. The conjugacy relation of ergodic measure preserving transformations is complete analytic.
The authors claim that this result finishes the classification problem of EMPT posed by von Neumann.
This is an instance where non classification is understood in terms of descriptive set theory.

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In fact this result is a corollary of a stronger statement about "Borel reducibility".

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Borel reducibility was first introduced in the late 80's by Friedman and Stanley in the context of model theory but it was quickly taken over by descriptive set theorists. The starting point comes from a generalization of theorems of Mackey, Glimm and Effros in operator algebras.

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Without a requirement on f, the definition would only amount to studying the cardinality of X/E vs. Y/F.

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Example 2 (Ornstein-Bowen): X Classical Bernoulli shifts, E conjugacy. $f(T) =$ the entropy of T.

The equivalence relation E_0

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Theorem (Baer): The isomorphism relation for countable rank 1 torsion free abelian groups is Borel bireducible to E_0.

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This really marks the beginning of Borel reducibility.

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3. $E_0 <_B E <_B E_\infty$ (Jackson, K, L)

Classification by countable structures

Definition. Let E be an equivalence relation on a Polish space X. E is *classifiable by countable structures* if

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Example. $\mathcal{L} = \{R, f\}$, R an n-ary relation symbol, and f an m-ary function symbol. Then

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Example II: (Halmos-vN) E = conjugacy of ergodic m.p. transformations with discrete spectrum. $\sigma_P(T)$ is a complete invariant. $E \leq_B E^Y_{S_\infty}$

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Theorem (S.-Törnquist, '08)

The isomorphism relation for separable von Neumann factors of type II_1, II_∞ *and* III_λ, $\lambda \in [0, 1]$, *are not classifiable by countable structures.*

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Theorem (S, '14):The isomorphism relation for separable McDuff factors of type II_1, is not classifiable by countable structures.

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A factor $M \in \mathrm{vN}(H)$ is *injective* (or *amenable* or *hyperfinite*) if it contains an increasing sequence of finite dimensional von Neumann algebras, with dense union in M. For each of the types $\mathrm{II}_1, \mathrm{II}_\infty$ and III_λ, $\lambda \in (0, 1]$, there is a unique injective factor of that type. However, for type III_0 we have:

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Borel Complete for countable structures

Denote by $\mathcal{F}_{\mathrm{II}_1}(\mathcal{H})$ the (standard) space of II_1 factors on $\mathcal{H}$, and by $\simeq^{\mathcal{F}_{\mathrm{II}_1}(\mathcal{H})}$ the isomorphism relation for factors of type II_1 on $\mathcal{H}$.

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As an immediate corollary, we have:

Corollary

The isomorphism relation for factors of type II_1 is complete analytic as a subset of $\mathcal{F}_{\mathrm{II}_1}(\mathcal{H}) \times \mathcal{F}_{\mathrm{II}_1}(\mathcal{H})$. In particular it is not a Borel subset.

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In other words: $\simeq^{\mathbf{wT}_{\mathrm{ICC}}}$ is *Borel complete* for countable structures, in the sense of Friedman and Stanley.